Sensitivity Analysis of an Inverse Problem for the Wave Equation with Caustics

Gang Bao* Hai Zhang†

November 28, 2012

Abstract

The paper investigates the sensitivity of the inverse problem of recovering the velocity field in a bounded domain from the boundary dynamic Dirichlet-to-Neumann map (DDtN) for the wave equation. Three main results are obtained: (1) assuming that two velocity fields are non-trapping and are equal to a constant near the boundary, it is shown that the two induced scattering relations must be identical if their corresponding DDtN maps are sufficiently close; (2) a geodesic X-ray transform operator with matrix-valued weight is introduced by linearizing the operator which associates each velocity field with its induced Hamiltonian flow. A selected set of geodesics whose conormal bundle can cover the cotangent space at an interior point is used to recover the singularity of the X-ray transformed function at the point; a local stability estimate is established for this case. Although fold caustics are allowed along these geodesics, it is required that these caustics contribute to a smoother term in the transform than the point itself. The existence of such a set of geodesics is guaranteed under some natural assumptions in dimension greater than or equal to three by the classification result on caustics and regularity theory of Fourier Integral Operators. The interior point with the above required set of geodesics is called "fold-regular"; (3) assuming that a background velocity field with every interior point fold-regular is fixed and another velocity field is sufficiently close to it and satisfies a certain orthogonality condition, it is shown that if the two corresponding DDtN maps are sufficiently close then they must be equal.

1 Introduction

This paper is concerned with the sensitivity (or stability) of the inverse problem of recovering the velocity field in a domain from the boundary dynamic Dirichlet-to-Neumann map (DDtN) in the wave equation. Let Ω be a bounded strictly convex smooth domain in $\mathbb{R}^d$, $d \geq 2$, with boundary Γ. Let $c(x)$ be a velocity field in Ω which characterizes the wave speed in the medium

*Department of Mathematics, Zhejiang University, Hangzhou 310027, China; Department of Mathematics, Michigan State University, East Lansing, MI 48824. The research was supported in part by the NSF grants DMS-0908325, DMS-0968360, DMS-1211292, the ONR grant N00014-12-1-0319, a Key Project of the Major Research Plan of NSFC (No. 91130004), and a special research grant from Zhejiang University. Email: **bao@math.msu.edu**

†Department of Mathematics, Michigan State University, East Lansing, MI 48824. Email: **zhangh20@msu.edu**

and let T be a sufficiently large positive number. We consider the following wave equation system:

$$\begin{aligned} \frac{1}{c^2(x)}u_{tt} - \Delta u &= 0, \quad (x,t) \in \mathbb{R}^d \times (0,T) & (1)\\ u(0,x) = u_t(0,x) &= 0, \quad x \in \Omega, & (2)\\ u(x,t) &= f(x,t), \quad (x,t) \in \Gamma \times (0,T). & (3) \end{aligned}$$

For each $f \in H_0^1([0,T] \times \Gamma)$, it is known that (see for instance [16]) there exists an unique solution $u \in C^1(0,T; L^2(\Omega)) \bigcap C(0,T; H^1(\Omega))$, and furthermore $\frac{\partial u}{\partial \nu} \in L^2([0,T] \times \Gamma)$, where ν is the unit outward normal to the boundary. The DDtN map Λ_c is defined by

$$\Lambda_c(f) := \frac{\partial u}{\partial \nu}|_{[0,T]\times\Gamma}.$$

The inverse problem is to recover the velocity function c from the DDtN map Λ_c. The uniqueness of the inverse problem is solved by the boundary control method first introduced by Belishev in [7]. The method can also be used to solve the uniqueness for more general problems, for instance, the anisotropic medium case. We refer to [8], [10], [9], [15] and the references therein for more discussions.

We are interested in the sensitivity question for the above inverse problem. Namely, we want to investigate how sensitive or stable is it to recover the velocity field from the DDtN map and characterize how a small change in the DDtN map affects the recovered velocity field.

The inverse problem of recovering velocity field is closely related to the inverse kinematic problem in geophysics, we refer to [21] for more discussions on the topic. It also can be viewed as a special case of the inverse problem of recovering a Riemannian metric on a Riemannian manifold. Indeed, it corresponds to the case when the metrics are restricted to the class of those which are conformal to the Euclidean one. The inverse problem of recovering a Riemannian metric has been extensively studied in the literature. The uniqueness is proved by Belishev and Kurylev in [10] by using the boundary control method. However, as pointed out in [26], their approach is unlikely to give a stability estimate since it uses in an essential way an unique continuation property of the wave equation.

The first stability result on the determination of the metric from the DDtN map was given by Stefanov and Uhlmann in [23], where they proved conditional stability of Hölder type for metrics close enough to the Euclidean one in C^k for $k \gg 1$ in three dimensions. Later, they extended the stability result to generic simple metrics, [26]. An important feature of their approach is to first derive a stability estimate of recovering the boundary distance function from the DDtN map and then apply existing results from the boundary rigidity problem in geometry. Their approach was extended by Montalto in [34] to study the more general problem of determine a metric, a co-vector and a potential simultaneously from the DDtN map, and a similar Hölder type conditional stability result was obtained. The stability of the inverse problem of determining the conformal factor to a fixed simple metric was studied by Bellassoued and Ferreira in [11]. They proved the Hölder type conditional stability result for the case when the conformal factors are close to one. We comment that the result in [11] holds for all simple metrics. For other stability results on the related problems, we refer to the references in [34].

We emphasize that all of the above stability results deal with the case when the metrics are simple. To our best knowledge, no stability result is available in the general case when

the metrics are not simple. This paper is devoted to the study of the general case when the metric induced by the velocity field is not simple. To avoid technical complications due to the boundary, we restrict our study to situation when the velocity fields are equal to one near the boundary. From this point of view, our results can be regarded as interior estimates. We refer to [26], [28] and the references therein for useful boundary estimates.

We now give a brief account of the approach and results in the paper. We first derive a sensitivity result of recovering the scattering relation from the DDtN map. Our result shows that two scattering relations must be identical if the two corresponding DDtN maps are sufficiently close in some suitable norm. Equivalently, any arbitrarily small change in the scattering relation can imply a certain change in the DDtN map. To our best knowledge, this seems to be the first sensitivity result for the problem in the non-simple metric case. Moreover, our result is fundamentally different from those in the literature where Lipschitz, Hölder or logarithmic estimates are derived , see for examples [6], [1], [31], [23], [24], [25], [26], [34] and [14]. This is the reason the term "sensitivity analysis" is used instead of "stability estimates" throughout the paper. We remark that when the geometry induced by the velocity field is simple, the scattering relation is equivalent to the boundary distance function. In that case, a Hölder type interior stability estimate for recovering the boundary distance function from DDtN map has been established in [26]. Compared to the Hölder type result, our result is much stronger. Our approach is based on Gaussian beam solutions to the wave equation, which are capable of dealing with caustics, major obstacles to the construction of classic geometric-optics solutions. We refer to [20] and [15] for more discussions on Gaussian beams and its applications.

We observe that for any velocity filed c, the induced Hamiltonian flow $\mathcal{H}_c^t$ when restricted to the unit cosphere bundle $S^*\mathbb{R}^d$ determines the scattering relation $\mathfrak{S}_c$. We linearize the operator which maps c to $\mathcal{H}_c^t|_{S^*\mathbb{R}^d}$ and obtain a geodesic X-ray transform operator $\mathfrak{I}_c$ with matrix-valued weight. Note that the scattering relation (or Hamiltonian flow) is the natural object to study when the metric is not simple. It is related to the lens rigidity problem in geometry. We refer to [30] for more discussions on the topic. The boundary distance function (global or local) has received extensive attention in the literature, whose has revealed many important results in the case of simple metrics and regular metrics, see for instance [25], [30]. However, this approach is unlikely to work in the case of general non-simple metrics. In this paper, we attempt to overcome the difficulty by analyzing the scattering relation (or Hamiltonian flow).

We study the inverse problem of recovering a vector-valued function f from its weighted geodesic transform $\mathfrak{I}_c f$. For a fixed interior point x, we use a carefully selected set of geodesics whose conormal bundle can cover the cotangent space $T_x^*\mathbb{R}^d$ to recover the singularity of f at x. We allow fold caustics along these geodesics, but require that these caustics contribute to a smoother term in the transform than x itself. It is still an open problem to show that such a set of geodesics exists generically for a general velocity field with caustics. But we draw evidence from the classification result on caustics and regularity theory of Fourier Integral Operators (FIOs) to show that it is the case under some natural assumptions in the dimensions equal or greater than three. We call the interior point with the above set of geodesics "fold-regular". A local stability estimate is derived near a fold-regular point.

Finally, we combine the stability result on the X-ray transform and the sensitivity result on recovering the scattering relation from the DDtN map to obtain a sensitivity result for the inverse problem.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we introduce the main results in the paper. Section 4 is devoted to the construction of Gaussian

beam solutions to the wave equation. The Gaussian beam solutions are used in Section 5 to prove the sensitivity result of determining scattering relation from DDtN map. In Section 6, we discuss the concept "fold-regular" and prove a local stability estimate for the geodesic X-ray transform $\mathfrak{I}_c$. In Section 7, we prove the sensitivity result of recovering the velocity field from the DDtN map.

Throughout, we use the following conventions:

1. Let f and g be two elements in a Hilbert space, then $\langle f, g\rangle$ stands for their inner product;
2. Let M_1 and M_2 be two matrices (including vectors which can be regarded as single column or single row matrices), then the product of M_1 and M_2 is denoted by $M_1 \cdot M_2$. Sometimes, the dot is omitted for simplicity;
3. Let M be a matrix, then $M^\dagger$ stands for its transpose. The same applies when M is a linear operator. If M is real and symmetric and C a real number, then $M \geq C$ means that the matrix $M - C \cdot Id$ is symmetric and positive definite. If M is a complex matrix, then we use $\Re M$ for its real part and $\Im M$ for its imaginary part;
4. Let U and V be two open set in a metric space, then $U \Subset V$ means that the closure of U, denoted by $\bar{U}$ is compact and is a subset of V;
5. Let C_1 and C_2 be two positive numbers, then $C_1 \lesssim C_2$ means that $C_1 \leq C \cdot C_2$ for some constant $C > 0$ independent of C_1 and C_2.

2 Preliminaries

In this section, we introduce some notations and definitions. Let Ω be a strictly convex smooth domain in $\mathbb{R}^d$ with boundary Γ. Let c be a smooth velocity field defined in Ω which is equal to one near the boundary. Then c has natural extension to $\mathbb{R}^d$. Throughout the paper, we always use the natural coordinate system of the cotangent bundle $T^*\mathbb{R}^d$ in which we write (x, ξ) for the co-vector $\xi_j dx^j$ in $T^*_x\mathbb{R}^d$. For ease of notation, we also use ξ for the co-vector $\xi_j dx^j$. The meaning of ξ should be clear from the context. The velocity field c introduces a Hamiltonian function $H_c(x,\xi) = \frac{1}{2}c^2(x)|\xi|^2$ to $T^*\mathbb{R}^d$. It also defines a norm to each cotangent space $T^*_x\mathbb{R}^d$ by

$$|\xi|_c = c(x)|\xi|, \quad \text{for} \;\; \xi \in T^*_x\mathbb{R}^d.$$

Throughout the paper, $|\cdot|$ stands for the usual Euclidean norm in $\mathbb{R}^d$, while $|\cdot|_c$ stands for the norm in $T^*\mathbb{R}^d$ induced by the function c. When there is no other velocity field in the context, we drop the subscript c and write $\|\cdot\|$ instead.

Denote the corresponding Hamiltonian flow by $\mathcal{H}_c^t$, i.e. for each $(x_0, \xi_0) \in T^*\mathbb{R}^d$, $\mathcal{H}_c^t(x_0,\xi_0) = (x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))$ solves the following equations:

$$\dot{x} \;=\; \frac{\partial H_c}{\partial \xi} = c^2 \cdot \xi, \quad x(0) = x_0, \tag{4}$$

$$\dot{\xi} \;=\; -\frac{\partial H_c}{\partial x} = -\frac{1}{2}\nabla c^2 \cdot |\xi|^2, \quad \xi(0) = \xi_0. \tag{5}$$

We call $(x(\cdot, x_0, \xi_0), \xi(\cdot, x_0, \xi_0))$ the bicharateristic curve emanating from (x_0, ξ_0) and $x(\cdot, x_0, \xi_0)$ the geodesic. By the assumptions on c, the flow $\mathcal{H}_c^t$ is defined for all $t \in \mathbb{R}$. Note that the flow $\mathcal{H}_c^t$ is also well-defined on the cosphere bundle $S^*\mathbb{R}^d = \{(x, \xi) : x \in \mathbb{R}, |\xi|_c = 1\}$.

We say that a velocity field c is non-trapping in Ω for time $T > 0$ if the following condition is satisfied:

$$\mathcal{H}_c^T(S^*\Omega) \bigcap S^*\Omega = \emptyset. \tag{6}$$

Denote

$$\begin{aligned} S_+^*\Gamma &= \{(x, \xi) : x \in \Gamma, |\xi|_c = 1, \langle \xi, \nu(x) \rangle > 0\}; \\ S_-^*\Gamma &= \{(x, \xi) : x \in \Gamma, |\xi|_c = 1, \langle \xi, \nu(x) \rangle < 0\}. \end{aligned}$$

Assume that the velocity field c is non-trapping in Ω for time T; we now define the scattering relation $\mathfrak{S}_c : S_-^*\Gamma \to S_+^*\Gamma$. For each $(x_0, \xi_0) \in S_-^*\Gamma$, let $l(x_0, \xi_0)$ be the first moment that the geodesic $x(\cdot, x_0, \xi_0)$ hits the boundary Γ. Define

$$\mathfrak{S}_c(x_0, \xi_0) = \mathcal{H}_c^{l(x_0,\xi_0)}(x_0, \xi_0).$$

For future reference, we define $l_- : S^*\Omega \to (-\infty, 0]$ by letting $l_-(x, \xi)$ be the first negative moment that the bicharacteristic curve $\mathcal{H}^t(x, \xi)$ hits the boundary $S_-^*\Gamma$ and $\tau : S^*\Omega \to S_-^*\Gamma$ by

$$\tau(x, \xi) = \mathcal{H}^{l_-(x,\xi)}(x, \xi).$$

We remark that $l_-(\cdot)$ and $\tau(\cdot)$ are well-defined by the assumption (6).

We now introduce the class of admissible velocity fields that are considered in the paper.

Definition 2.1. *Let M_0, ϵ_0 and T be positive numbers. A velocity field c is said to belong to the admissible class $\mathfrak{A}(M_0, \epsilon_0, \Omega, T)$ if and only if the following three conditions are satisfied:*

1. *$c \in C^3(\mathbb{R}^d)$, $0 < \frac{1}{M_0} \le c \le M_0$, and $\|c\|_{C^3(\mathbb{R}^d)} \le M_0$;*
2. *the support of $c - 1$ is contained in the set $\Omega_{\epsilon_0} =: \{x \in \Omega : dist(x, \Gamma) < \epsilon_0\}$;*
3. *the Hamiltonian H_c is non-trapping in Ω for time T.*

By Condition 2 above, we can find two small positive constants ϵ^* and ϵ_1, both depending on ϵ_0, such that for any $(x_0, \xi_0) \in S_-^*\Gamma$, if $\{\mathcal{H}^t(x_0, \xi_0) : t \in (0, l(x_0, \xi_0))\} \bigcap S^*\Omega_{\epsilon_0} \neq \emptyset$, then

$$\begin{aligned} \langle \xi_0, \nu(x_0) \rangle &\le -\epsilon^*, && (7) \\ \langle \xi_1, \nu(x_1) \rangle &\ge \epsilon^*, && (8) \\ l(x_0, \xi_0) &\ge \epsilon_1, && (9) \end{aligned}$$

where $(x_1, \xi_1) = \mathfrak{S}_c(x_0, \xi_0)$.

Finally, we remark that we set up the discussion in the paper in the cotangent space $T^*\mathbb{R}^d$. But one can also set up the discussion in the tangent space $T\mathbb{R}^d$, see for instance [22], [30]. The equivalence of the two setups can be seen from the procedure of "raising and lowing indices" in Riemannian geometry. We choose the cotangent setup mainly because the following three reasons. First, it is more natural to the construction of Gaussian beams. Second, the classification result of singular Lagrangian maps is more complete than that of singular exponential maps in

the literature, though these two problems are equivalent in Riemannian manifold. Finally, it is more natural to study caustics in the cotangent space.

3 Statement of the main results

3.1 Sensitivity of recovering the scattering relation from the DDtN map

It is known that the DDtN map Λ_c determines the scattering relation $\mathfrak{S}_c$ uniquely [19]. We show that the following sensitivity result of recovering the scattering relation from the DDtN map holds. The proof is given in Section 5.

Theorem 3.1. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$. Then there exists a constant $\delta > 0$ such that*

$$\mathfrak{S}_{\tilde{c}} = \mathfrak{S}_c$$

if $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$.

Remark 3.1. *The same result holds when the velocity fields are replaced by symmetric positive definite matrices.*

3.2 Linearization of the operator which maps velocity fields to Hamiltonian flows

We begin with the following observation.

Lemma 3.1. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$, then $\mathfrak{S}_c = \mathfrak{S}_{\tilde{c}}$ if and only if $\mathcal{H}_c^T|_{S_-^*\Gamma} = \mathcal{H}_{\tilde{c}}^T|_{S_-^*\Gamma}$.*

The above lemma shows the equivalence of the Hamiltonian flow and the scattering relation. The next lemma shows that $\mathcal{H}_c^t$ satisfies an equivalent ordinary differential equation (ODE) system in $S^*\mathbb{R}^d$.

Lemma 3.2. *Let $(x_0, \xi_0) \in S^*\mathbb{R}^d = \{(x, \xi) \in \mathbb{R}^{2d} : c(x)|\xi| = 1\}$, and let $(x(t), \xi(t)) = \mathcal{H}_c^t(x_0, \xi_0)$, then $(x(t), \xi(t))$ satisfies the following ODE system*

$$\dot{x} = \frac{\xi}{|\xi|^2}, \tag{10}$$

$$\dot{\xi} = b(x). \tag{11}$$

where $b(x) = -\frac{1}{2}\nabla \ln c^2$. Conversely, if $(x(t), \xi(t)) \in S^*\mathbb{R}^d$ satisfies the ODE system (10)-(11), then $(x(t), \xi(t)) = \mathcal{H}_c^t(x_0, \xi_0)$.

We next linearize the operator which maps each velocity field to its induced Hamiltonian flow restricted to the cosphere bundle. Let c be a fixed smooth background velocity field. Denote the perturbed velocity field and Hamiltonian flow at time T as $\tilde{c}^2 = c^2 + \delta c^2$ and $\mathcal{H}_{\tilde{c}}^T = \mathcal{H}_c^T + \delta\mathcal{H}_c^T$ respectively. Denote also that $\delta b = -\frac{1}{2}\nabla(\ln \tilde{c}^2 - \ln c^2)$ and

$$A(x, \xi) = \begin{pmatrix} 0 & \frac{\partial}{\partial \xi}\left(\frac{\xi}{|\xi|^2}\right) \\ \frac{\partial b}{\partial x} & 0 \end{pmatrix}.$$

For each $(x_0, \xi_0) \in S^*_-\Gamma$, let $\Phi(t, x_0, \xi_0)$ be the solution of the following ODE system

$$\dot{\Phi}(t) = -\Phi(t) A(\mathcal{H}_c^t), \quad \Phi(0) = Id.$$

By the results in Appendix 8.1, we have

$$\delta\mathcal{H}_c^T = \frac{\delta\mathcal{H}_c^T}{\delta b}(\delta b) + r(\delta b),$$

where

$$\frac{\delta\mathcal{H}_c^T}{\delta b}(\delta b)(x_0, \xi_0) = \int_0^T \Phi^{-1}(T, x_0, \xi_0) \cdot \Phi(s, x_0, \xi_0) \begin{pmatrix} 0 \\ b(x(s, x_0, \xi_0)) \end{pmatrix} ds \tag{12}$$

and $\|r(\delta b)\|_{L^\infty} \le C\|\delta b\|_{C^1}^2$ for some constant $C > 0$ depending only on $\|c\|_{C^3(\mathbb{R}^d)}$.

Formula (12) motivates us to define the following geodesic X-ray transform operator

$$\mathfrak{I}_c(f)(x_0, \xi_0) = \int_0^T \Phi(s, x_0, \xi_0) f(x(s, x_0, \xi_0))\, ds, \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d}). \tag{13}$$

Then $\frac{\delta\mathcal{H}_c^T}{\delta b}(\delta b)(x_0, \xi_0) = \Phi^{-1}(T, x_0, \xi_0) \cdot \mathfrak{I}_c(f)(x_0, \xi_0)$ with

$$f = \begin{pmatrix} 0 \\ \frac{1}{2}\nabla(\ln c^2 - \ln \tilde{c}^2) \end{pmatrix}. \tag{14}$$

We associate each $(x, \xi) \in S^*\Omega$ a matrix $\Phi(x, \xi)$. Let $(x_0, \xi_0) = \tau(x, \xi) = \mathcal{H}_c^{l_-(x,\xi)}(x, \xi)$. We then define

$$\Phi(x, \xi) = \Phi(-l_-(x, \xi), \tau(x, \xi)).$$

It is clear that the following identity holds

$$\Phi(\mathcal{H}_c^s(x_0, \xi_0)) = \Phi(s, x_0, \xi_0)$$

for all $s \in \mathbb{R}_+$ such that $\mathcal{H}_c^s(x_0, \xi_0) \in S^*\Omega$. We can rewrite the X-ray transform operator $\mathfrak{I}_c$ in the following standard form

$$\begin{aligned} \mathfrak{I}_c f(x_0, \xi_0) &= \int_0^T \Phi(\mathcal{H}_c^s(x_0, \xi_0)) f(\pi(\mathcal{H}_c^s(x_0, \xi_0)))\, ds \\ &= \int_0^{l(x_0,\xi_0)} \Phi(\mathcal{H}_c^s(x_0, \xi_0)) f(\pi(\mathcal{H}_c^s(x_0, \xi_0)))\, ds. \end{aligned} \tag{15}$$

Remark 3.2. *Formula (15) is derived in the coordinate of $T^*\mathbb{R}^d$. Hence it may not be geometrically invariant.*

Lemma 3.3. *Assume that $\mathfrak{S}_c = \mathfrak{S}_{\tilde{c}}$, let f be defined as in (14), then*

$$\|\mathfrak{I}_c f\|_{L^\infty} \lesssim \|f\|_{C^1(\Omega)}^2.$$

3.3 Fold-regular points and local stability for geodesic X-ray transform

We consider the stability estimate of the operator $\mathfrak{I}_c$. For simplicity, we drop the subscript c. Define $\beta : T^*R^d \backslash \{(x, 0) : x \in R^d\} \to S^*\mathbb{R}^d$ by

$$\beta(x, \xi) = \left(x, \frac{\xi}{\|\xi\|}\right).$$

Let $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ be the natural projection onto the base space. We define $\phi : T^*\mathbb{R}^d \to \mathbb{R}^d$ by

$$\phi(x, \xi) = \pi \circ \mathcal{H}^{t=1}(x, \xi), \quad (x, \xi) \in T^*\mathbb{R}^d.$$

We remark that ϕ defined above is equivalent to the exponential map in Riemannian manifold.

The following result about the normal operator $\mathfrak{N} = \mathfrak{I}^\dagger \mathfrak{I}$ is well-known.

Lemma 3.4. *The normal operator $\mathfrak{N} : L^2(\Omega, \mathbb{R}^{2d}) \to L^2(\Omega, \mathbb{R}^{2d})$ is bounded and has the following representation*

$$\mathfrak{N}f(x) = \int_{T_x^*\Omega} W(x, \xi) f(\phi(x, \xi))\, d\sigma_x(\xi), \quad f \in L^2(\Omega, \mathbb{R}^{2d}) \tag{16}$$

where $d\sigma_x$ denotes the measure in the space $T_x^*\mathbb{R}^d$ induced by the velocity field c, i.e. $d\sigma_x(\xi) = c(x)^d d\xi$, and W is defined as

$$W(x, \xi) = \frac{1}{\|\xi\|^{d-1}} \{\Phi^\dagger \circ \beta(x, \xi) \cdot \Phi \circ \beta \circ \mathcal{H}(x, \xi) + \Phi^\dagger \circ \beta(x, -\xi) \cdot \Phi \circ \beta \circ \mathcal{H}^{-1}(x, -\xi)\}. \tag{17}$$

Proof. See [33] or [24].

We see from (16) that the local property of the normal operator $\mathfrak{N}$ restricted to a small neighborhood of $x \in \Omega$ is determined by the lagrangian map $\phi(x, \cdot) : T_x^*\mathbb{R}^d \to \mathbb{R}^d$. When the map is a diffeomorphism, it is known that the operator $\mathfrak{N}$ near x is a pseudo-differential operator (ΨDO). However, in general case, the map may not be a diffeomorphism and may have singular points which are called caustic vectors. The value of the map at caustic vectors are called caustics. When caustics occur, the Schwartz kernel of the operator $\mathfrak{N}$ has two singularities, one is from the diagonal which contributes to a ΨDO $\mathfrak{N}_1$, and the other is from the caustics which contributes to a singular integral operator $\mathfrak{N}_2$. The property of $\mathfrak{N}_2$ depends on the type of caustics. The case for fold caustics is investigated in [33], where it is shown that fold caustics contribute a Fourier Integral Operator (FIO) to $\mathfrak{N}_2$. Little is known for caustics of other type. Here we recall the following definition of fold caustics.

Definition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a germ of C^∞ map at x_0, then x_0 is said to be a fold vector and $f(x_0)$ a fold caustic if the following two conditions are satisfied:*

1. *the rank of df at x_0 equals to $n-1$ and $\det df$ vanishes of order 1 at x_0;*
2. *the kernel of the matrix $df(x_0)$ is transversal to the manifold $\{x : \det df(x) = 0\}$ at x_0.*

We now introduce the following concept of "operator germ" to characterize the contribution of an infinitesimal neighborhood of a caustic or a regular point to the normal operator $\mathfrak{N}$.

Definition 3.2. *For each $\xi \in T_x^*\mathbb{R}^d\backslash 0$, the operator germ $\mathfrak{N}_\xi$ is defined to be the equivalent class of operators in the following form*

$$\mathfrak{N}_\xi f(y) = \int_{T_y^*\Omega} W(y,\eta) f(\phi(y,\eta))\chi(y,\eta)\, d\sigma_y(\eta). \tag{18}$$

where χ is a smooth function supported in a small neighborhood of (x,ξ) in $\mathbb{R}^{2d}$. Two operators with χ_1 and χ_2 are said to be equivalent if there exists a neighborhood $B(x,\xi)$ of (x,ξ) such that $\chi_1 = \chi_2 \cdot \chi_3$ for some $\chi_3 \in C_0^\infty(B(x,\xi))$ with $\chi_3(x,\xi) \neq 0$.

The operator germ $\mathfrak{N}_\xi$ is said to has certain property if there exists a neighborhood $B(x,\xi)$ of (x,ξ) in $T^*\mathbb{R}^d$ such that the property holds for all operators of the form (18) with $\chi \in C_0^\infty(B(x,\xi))$.

Properties of the above defined operator germ will be given in Section 6.1.

We note from the preceding discussion that it is complicated to analyze the full operator $\mathfrak{N}$ which contains information from all geodesics. However, for a given interior point x, to recover f or the singularity of f at x from its geodesic transform, we need only to select a set of geodesics whose conormal bundle can cover the cotangent space $T_x^*\mathbb{R}^d$. Caustics may be allowed along these geodesics as long as they are of the simplest type, i.e fold type so that we can analyze their contributions. This idea can be carried out by introducing a cut-off function for the set of geodesics as we do now. We remark that this idea is motivated by the work [28]. For any $\alpha \in C_0^\infty(S_-^*\Gamma)$, we define

$$\mathfrak{I}_\alpha f(x_0,\xi_0) = \alpha(x_0,\xi_0) \int_0^{l(x_0,\xi_0)} \Phi(\mathcal{H}_c^s(x_0,\xi_0)) f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\, ds \tag{19}$$

where $(x_0,\xi_0) \in S_-^*\Gamma$. Let $\alpha^\sharp$ be the unique lift of α to $S^*\Omega$ which is constant along bicharacteristic curves, i.e. $\alpha^\sharp(x,\xi) = \alpha \circ \tau(x,\xi)$ for $(x,\xi) \in S^*\Omega$. Then $\alpha^\sharp$ is smooth in $S^*\Omega$ and we have

$$\mathfrak{I}_\alpha f(x_0,\xi_0) = \int_0^{l(x_0,\xi_0)} (\alpha^\sharp \cdot \Phi)(\mathcal{H}_c^s(x_0,\xi_0)) f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\, ds \tag{20}$$

With the original weight Φ being replaced by the new one $\alpha^\sharp \cdot \Phi$, we similarly can define $\mathfrak{N}_\alpha$. In fact, it is easy to check that $\mathfrak{N}_\alpha$ is defined as in (17) with W being replaced by

$$\begin{aligned} W_\alpha(x,\xi) &= \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,\xi)|^2 \Phi^\dagger \circ \beta(x,\xi) \cdot \Phi \circ \beta \circ \mathcal{H}(x,\xi) \\ &\quad + \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,-\xi)|^2 \Phi^\dagger \circ \beta(x,-\xi) \cdot \Phi \circ \beta \circ \mathcal{H}^{-1}(x,-\xi). \end{aligned}$$

It can be shown that with properly chosen α, the analysis of the operator $\mathfrak{N}_\alpha$ becomes possible and we can recover the singularity of f from $\mathfrak{N}_\alpha f$.

We now give two definitions whose discussions are postponed to Section 6.

Definition 3.3. *A fold vector $\xi \in T_x^*\mathbb{R}^d$ is called fold-regular if there exists a neighborhood $U(x)$ of x such that the operator germ $\mathfrak{N}_\xi$ is compact from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^1(U(x), \mathbb{R}^{2d})$ (or from $H^s(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^{s+1}(U(x), \mathbb{R}^{2d})$ for all $s \in \mathbb{R}$).*

Definition 3.4. *A point x is called fold-regular if there exists a compact subset $\mathcal{Z}_2(x) \subset S_x^*\mathbb{R}^d$ such that the following two conditions are satisfied:*

1. *For each $\xi \in \mathcal{Z}_2(x)$, there exist only singular vectors of fold-regular type along the ray $\{t\xi : t \in \mathbb{R}\}$ for the map $\phi(x, \cdot)$;*

2. *$\forall \xi \in S^*_x\mathbb{R}^d$, $\exists \theta \in \mathcal{Z}_2(x)$, such that $\theta \perp \xi$.*

We remark that $\mathcal{Z}_2(x)$ parameterizes a subset of geodesics that pass through x and along which there exist only fold-regular caustics.

We now present the main result on the local stability estimate for the geodesic X-ray transform operator. The proof is given in Section 6.

Theorem 3.2. *Let x_* be a fold-regular point, then there exist a cut-off function $\alpha \in C_0^\infty(S^*_-\Gamma)$, a neighborhood $U(x_*)$ of x_*, a compact operator $\mathfrak{N}_{2,\alpha}$ from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^1(U(x_*), \mathbb{R}^{2d})$ and a smoothing operator $\mathfrak{R}$ from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$, such that for any $U_0(x_*) \Subset U(x_*)$ the following holds*

$$\|f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} \lesssim \|\mathfrak{N}_\alpha f\|_{H^{s+1}(U(x_*),\mathbb{R}^{2d})} + \|\mathfrak{N}_{2,\alpha} f\|_{H^{s+1}(U(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U(x_*),\mathbb{R}^{2d})} \quad (21)$$

for all $f \in \mathcal{D}'(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ and $s \in \mathbb{R}$.

3.4 Sensitivity of recovering the velocity field from the DDtN map

Definition 3.5. *An admissible velocity field c is called fold-regular if all points in Ω are fold-regular with respect to the Hamiltonian flow $\mathcal{H}^t_c$.*

We have established the following main result on the sensitivity of recovering velocity field from DDtN map. For simplicity we only consider the case $d = 3$, similar results also hold for $d > 3$. The proof is given in Section 7.

Theorem 3.3. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$. Assume that the velocity field c is smooth and is fold-regular. Then there exist a finite dimensional subspace $\mathfrak{L} \subset L^2(\Omega_{\epsilon_0}, \mathbb{R}^3)$, and a constant $\delta > 0$ such that for all $\tilde{c}$ sufficiently close to c in $H^{\frac{17}{2}}(\Omega)$ and satisfying $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$, $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$ implies that $c = \tilde{c}$.*

4 Gaussian beam solutions to the wave equation

Let c be a velocity field in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$. We construct Gaussian beam solutions to the wave equation system (1)-(3) in this section.

We first construct a Gaussian beam in $\mathbb{R}^d$. Following [18], we define $G(x, \xi) = c(x)|\xi|$. For a given $(x_0, \xi_0) \in S^*_-\Gamma$, let $(x(t), \xi(t), M(t), a(t))$ be the solution to the following ODE system:

$$\begin{aligned}
\dot{x} &= G_p, \qquad x(t_0) = x_0, \\
\dot{\xi} &= -G_x, \qquad \xi(t_0) = \xi_0, \\
\dot{M} &= -G^\dagger_{x\xi} M - M G_{\xi x} - M G_{\xi\xi} M - G_{xx}, \quad M(t_0) = \sqrt{-1} \cdot Id, \qquad (22) \\
\dot{a} &= -\frac{a}{2G}(c^2 \text{trace}(M) - G^\dagger_x G_\xi - G^\dagger_\xi M G_\xi), \quad a(t_0) = \lambda^{\frac{d}{4}}. \qquad (23)
\end{aligned}$$

The corresponding Gaussian beam with frequency λ ($\lambda \gg 1$) is given as follows

$$g(t, x, \lambda) = a(t) e^{i\lambda\tau(t,x)}$$

where $\tau(t,x) = \xi(t)\cdot(x-x(t)) + \frac{1}{2}(x-x(t))^\dagger M(t)(x-x(t))$.

Now, let the beam g impinge on the surface Γ, we want to construct the reflected beam g^-. Without loss of generality, we may assume that the ray $x(t)$ hits Γ at the point $x(t_1) = x_1$. Write $\xi(t_1) = \xi_1$. We parameterize Γ in a neighborhood of x_1, say $V(x_1)$, by a smooth diffeomorphism $F : U(x_1) \to V(x_1)$, where $U(x_1)$ is a neighborhood of the origin in $\mathbb{R}^{d-1}$. We require that $F(0) = x_1$. With the coordinate $x = F(y)$, we can rewrite functions restricted to the boundary Γ. For example, we rewrite

$$g(t,x) = g(t,F(y)) = \hat{g}(t,y), \quad \tau(t,x) = \tau(t,F(y)) = \hat{\tau}(t,y), \quad \text{for } x \in V(x_1).$$

We next derive formulas for $\hat{\tau}(t,y)$ and $\hat{g}(t,y)$. For this, we need to calculate $\hat{\tau}(t_1,0)$, $\frac{\partial\hat{\tau}}{\partial t}(t_1,0)$, $\frac{\partial\hat{\tau}}{\partial y}(t_1,0)$ and $\hat{M}(t_1) =: \frac{\partial^2\hat{\tau}}{\partial(t,y)^2}(t_1,0)$. In fact, by direct calculation, we have

$$\frac{\partial\hat{\tau}}{\partial t}(t_1,0) = -1, \quad \frac{\partial\hat{\tau}}{\partial y}(t_1,0) = (\frac{\partial F}{\partial y}(0))^\dagger\xi_1.$$

Moreover, the imaginary part and real part of the matrix $\frac{\partial^2\hat{\tau}}{\partial(t,y)^2}(t_1,0)$ are given below

$$\begin{aligned}
\Im\hat{M}(t_1) &= \begin{pmatrix} c^4(x_1)\xi_1^\dagger\Im M(t_1)\xi_1 & -c^2(x_1)\xi_1^\dagger\Im M(t_1)\cdot\frac{\partial F}{\partial y}(0) \\ -c^2(x_1)(\frac{\partial F}{\partial y}(0))^\dagger\Im M(t_1)\xi_1 & (\frac{\partial F}{\partial y}(0))^\dagger\Im M(t_1)\frac{\partial F}{\partial y}(0) \end{pmatrix} \\
&= R(t_1,x_1)^\dagger\Im M(t_1)R(t_1,x_1), \\
\Re\hat{M}(t_1) &= R(t_1,x_1)^\dagger\Re M(t_1)R(t_1,x_1) + \begin{pmatrix} \frac{1}{2}\nabla c^2(x)\xi_1 & -(\nabla\ln c(x_1))^\dagger\frac{\partial F}{\partial y}(0) \\ -(\frac{\partial F}{\partial y}(0))^\dagger\nabla\ln c(x_1) & \frac{\partial^2 F}{\partial y^2}(0)\xi_1 \end{pmatrix},
\end{aligned}$$

where $R(t_1,x_1) = (c^2(x_1)\xi_1, \frac{\partial F}{\partial y}(0))$.

We claim that

$$\Im\hat{M}(t_1) > 0.$$

Indeed, note that the column vectors in the matrix $\frac{\partial F}{\partial y}(0)$ are linearly independent and hence span the tangent space of the surface Γ at the point x_1. By (8), ξ_1 forms a nonzero angle with the tangent space and thus is linearly independent with all the column vectors in the matrix $\frac{\partial F}{\partial y}(0)$. Therefore the matrix $R(t_1,x_1)$ is invertible, and our claim follows. Using (7) and (8), we further have

$$\Im\hat{M}(t_1) > C \tag{24}$$

for some $C > 0$ depending on ϵ_0 and M_0.

Now, we have calculated $\hat{\tau}(t_1,0)$, $\frac{\partial\hat{\tau}}{\partial t}(t_1,0)$, $\frac{\partial\hat{\tau}}{\partial y}(t_1,0)$ and $\frac{\partial^2\hat{\tau}}{\partial(t,y)^2}(t_1,0)$. It follows that

$$\begin{aligned}
\hat{\tau}(t_1,y) &= \hat{\tau}(t_1,0) + (\frac{\partial\hat{\tau}}{\partial(t,y)}(t_1,0))^\dagger(t-t_1,y) + (t-t_1,y)\frac{\partial^2\hat{\tau}}{\partial(t,y)^2}(t_1,0)(t-t_1,y)^\dagger \\
&\quad +O(|(t-t_1,y)|^3) \\
&= \langle(-1,\frac{\partial F}{\partial y}(0)^\dagger\xi_1),(t-t_1,y)\rangle + (t-t_1,y)\hat{M}(t_1)(t-t_1,y)^\dagger + O(|(t-t_1,y)|^3).
\end{aligned}$$

We proceed to construct the reflected beam g^-. Write

$$g^-(t,x,\lambda) = a^-(t)e^{i\lambda\tau^-(t,x)}$$

with

$$\tau^-(t,x) = \xi^-(t)\cdot(x - x^-(t)) + \frac{1}{2}(x - x^-(t))^\dagger M^-(t)(x - x^-(t)).$$

We need to find $(x^-(t_1), \xi^-(t_1), a^-(t_1), M^-(t_1))$ such that the $g^- + g \approx 0$ on the boundary. Following [2], we impose the following condition

$$\partial_{t,y}^\alpha \hat{\tau}(t_1,0) = \partial_{t,y}^\alpha \hat{\tau}^-(t_1,0), \quad \text{for all } |\alpha| \le 2. \tag{25}$$

The above condition with $|\alpha| = 0$ gives that $\hat{\tau}^-(t_1,0) = \hat{\tau}(t_1,0) = 0$; with $|\alpha| = 1$ gives that

$$(\frac{\partial F}{\partial y}(0))^\dagger \xi_1 = (\frac{\partial F}{\partial y}(0))^\dagger \xi_1^-, \tag{26}$$

where $\xi_1^- = \xi^-(t_1)$. Since the column vectors in $(\frac{\partial F}{\partial y}(0))^\dagger$ spans the tangent space $T_{x_1}\Gamma$, we see that the tangential component of ξ_1^- and ξ_1 are equal. Besides, note that $|\xi_1| = |\xi_1^-| = 1$. Thus,

$$\xi^-(t_1) = \xi_1^- = \xi_1 - 2\langle \xi_1, \nu(x_1)\rangle \nu(x_1).$$

Condition (25) with $|\alpha| = 2$ gives that $\hat{M}^-(t_1) = \hat{M}(t_1)$. Recall the relation between $\Im M^-(t_1)$ and $\Im M(t_1)$, $\Re M^-(t_1)$ and $\Re M(t_1)$, we have the following two identities:

$$\begin{aligned}
R(t_1,x_1)^\dagger \Im M(t_1) R(t_1,x_1) &= R(t_1,x_1)^\dagger \Im M^-(t_1) R(t_1,x_1), \\
R(t_1,x_1)^\dagger \Re M(t_1) R(t_1,x_1) &= R(t_1,x_1)^\dagger \Im M^-(t_1) R(t_1,x_1) \\
&\quad + \begin{pmatrix} -\frac{1}{2}\nabla c^2(x)(\xi_1^- - \xi_1) & 0 \\ 0 & \frac{\partial^2 F}{\partial y^2}(0)(\xi_1^- - \xi_1) \end{pmatrix}.
\end{aligned}$$

Solving the above equations, we obtain $\Im M^-(t_1)$ and $\Re M^-(t_1,)$ and hence $M^-(t_1)$. Finally, set $a^-(t_1) = -a(t_1)$. Then all of the four components of $(x^-(t_1), \xi^-(t_1), a^-(t_1), M^-(t_1))$ are constructed. We then solve an ODE system to get $(x^-(t), \xi^-(t), M^-(t), a^-(t))$ as we did for the beam g. This completes the construction for the reflected beam g^-.

We now present some properties about the constructed beam. The following lemma is crucial in the subsequent estimates. We refer to [20] for the proof.

Lemma 4.1. *Both the matrices $M(t)$ and $M^-(t)$ are uniformly bounded for $t \in [0, T+\epsilon_1]$. Moreover, there exists $C > 0$, depending on M_0 and ϵ_0, such that $\Im M(t) > C$ and $\Im M^-(t) > C$ for all $t \in [0, T+\epsilon_1]$.*

We next introduce two auxiliary beams below

$$\hat{g}_*(t,y,\lambda) = a(t_1)e^{i\lambda\hat{\tau}_*}, \quad \hat{g}_*^-(t,y,\lambda) = a^-(t_1)e^{i\lambda\hat{\tau}_*^-},$$

where

$$
\begin{aligned}
\hat{\tau}_* &= \left\langle(-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1), (t-t_1, y)\right\rangle + (t-t_1, y)\hat{M}(t_1, 0, 0)(t-t_1, y)^\dagger, \\
\hat{\tau}_*^- &= \left\langle(-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1^-), (t-t_1, y)\right\rangle + (t-t_1, y)\hat{M}^-(t_1, 0, 0)(t-t_1, y)^\dagger.
\end{aligned}
$$

It is clear that $\hat{\tau}_* = \hat{\tau}_*^-$ and $\hat{g}_* = -\hat{g}_*^-$.

Lemma 4.2.

$$
\begin{aligned}
\hat{g}(t, y, \lambda) &= \hat{g}_*(t, y, \lambda) + O(\sqrt{\lambda}) \quad in \ \ H^1((3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)), && (27) \\
\hat{g}^-(t, y, \lambda) &= \hat{g}_*^-(t, y, \lambda) + O(\sqrt{\lambda}) \quad in \ \ H^1((3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)). && (28)
\end{aligned}
$$

Proof: See Appendix 8.2.

Note that $\|\hat{g}(t, y, \lambda)\|_{L^2((t_1-\epsilon_1/2, t_1+\epsilon_1/2)\times U(x_1))} \approx 1$. As a direct consequence of Lemma 4.2, we obtain the following norm estimate for the beam g restricted to the boundary Γ.

Lemma 4.3.

$$
\|g(\cdot, \cdot, \lambda)\|_{L^2((t_1-\epsilon_1/2, t_1+\epsilon_1/2)\times V(x_1))} \approx 1. \tag{29}
$$

We now present an H^1-norm estimate for $g^- + g$ and an approximation for the Neumann data $\frac{\partial g}{\partial \nu}^- + \frac{\partial g}{\partial \nu}$ on the boundary.

Lemma 4.4.

$$
\begin{aligned}
g^-(t, x, \lambda) + g(t, x, \lambda) &= O(\sqrt{\lambda}) \quad in \ \ H^1((3\epsilon_1/4, t_1 + \epsilon_1/2) \times V(x_1)); && (30) \\
\frac{\partial g}{\partial \nu}^- + \frac{\partial g}{\partial \nu} &= 2i\lambda g \cdot \langle \xi_1, \nu(x_1)\rangle + O(\sqrt{\lambda}) \quad in \ \ L^2((3\epsilon_1/4, t_1 + \epsilon_1/2) \times V(x_1)) && (31)
\end{aligned}
$$

Proof: See Appendix 8.2.

Now, we are ready to construct Gaussian beam solutions to the initial boundary value problem of the wave system (1)-(3). We first choose $\chi_{\epsilon_1}(t) \in C_0^\infty(\mathbb{R})$ such that $\chi_{\epsilon_1}(t) = 1$ for $t \in (\epsilon_1/4, \epsilon_1/2)$ and $\chi_{\epsilon_1}(t) = 0$ for $t \in (-\infty, 0)\bigcup(3\epsilon_1/4, \infty)$. Let $(x_0, \xi_0) \in S_-^*\Gamma$ and $(x_0^*, \xi_0^*) = \mathcal{H}^{-\frac{\epsilon_1}{4}}(x_0, \xi_0) = (x_0 - \frac{\epsilon_1 \cdot \xi_1}{4}, \xi_0)$. Let g be the Gaussian beam constructed with the initial data $x(0) = x_0^*, \xi(0) = \xi_0^*, M(0) = i \cdot Id$ and $a(0) = \lambda^{\frac{d}{4}}$. The beam g is reflected by Γ at $(x_1, \xi_1) = \mathfrak{S}_c(x_0, \xi_0) = \mathcal{H}_c^{l(x_0,\xi_0)}(x_0, \xi_0)$ at $t_1 = l(x_0, \xi_0) + \frac{\epsilon_1}{4}$. We construct the reflected beam g^- by the preceding procedure. Let u be the exact solution to the wave system (1)-(3) with

$$
f(t, x, \lambda) = g(t, x, \lambda) \cdot \chi_{\epsilon_1}(t).
$$

Then $u = g + g^- + R$, where the remaining term R satisfies the following equation system

$$
\begin{aligned}
\mathcal{P}R &= -\mathcal{P}(g + g^-), \quad (t, x) \in \Omega \times (0, t_1 + \epsilon_1/2), \\
R(0, x, \lambda) &= -(g + g^-)(0, x, \lambda), \quad x \in \Omega, \\
R_t(0, x, \lambda) &= -(g_t + g_t^-)(0, x, \lambda), \quad x \in \Omega, \\
R(t, x, \lambda) &= -g(t, x, \lambda)(1 - \chi_{\epsilon_1}(t)) - g^-(t, x, \lambda), \quad (t, x) \in (0, t_1 + \epsilon_1/2) \times \Gamma.
\end{aligned}
$$

Here $\mathcal{P}$ stands for the wave operator $\frac{1}{c^2(x)}\partial_{tt} - \Delta$.

Lemma 4.5.

$$\|\frac{\partial R}{\partial \nu}\|_{L^2([0,t_1+\epsilon_1/2]\times\Gamma)} \leq C\sqrt{\lambda}$$

for some constant $C > 0$ *depending on* ϵ_0 *and* M_0*.*

Proof. We apply Theorem 4.1 in [16] to derive the estimate. Note that the compatibility condition is satisfied on the boundary at time $t = 0$. It remains to show that the following four estimates hold:

$$\begin{aligned}
\|\mathcal{P}(g + g^-)\|_{C([0,t_1+\epsilon_1/2];L^2(\Omega))} &\lesssim \sqrt{\lambda}, &(32)\\
\|(g + g^-)(0,\cdot,\lambda)\|_{H^1(\Omega)} &\lesssim \sqrt{\lambda}, &(33)\\
\|(g_t + g_t^-)(0,\cdot,\lambda)\|_{L^2(\Omega)} &\lesssim \sqrt{\lambda}, &(34)\\
\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1([0,t_1+\epsilon_1/2]\times\Gamma)} &\lesssim \sqrt{\lambda}. &(35)
\end{aligned}$$

First, (32) follows from the standard estimate for Gaussian beams, see for example [5]. We next show (33). By Lemma 4.1, there exists a constant $C > 0$ depending on M_0 and ϵ_0 such that the following two inequalities hold

$$\begin{aligned}
|g(t,x,\lambda)| &\lesssim \lambda^{\frac{d}{4}} \cdot e^{-C\lambda\cdot|x-x^-(t)|^2},\\
|g^-(t,x,\lambda)| &\lesssim \lambda^{\frac{d}{4}} \cdot e^{-C\lambda\cdot|x-x^-(t)|^2}.
\end{aligned}$$

Thus the beam g and g^- are exponentially decaying away from the ray $x(t)$ and $x^-(t)$ respectively. Using this property, it is straightforward to show that $\|g(0,\cdot,\lambda)\|_{H^1(\Omega)} \lesssim 1$ and $\|g^-(0,\cdot,\lambda)\|_{H^1(\Omega)} \lesssim 1$, whence (33) and (34) follows.

Now, we show (35). We divide the domain $(0, t_1 + \epsilon_1/2) \times \Gamma$ into three parts:

$$\Sigma_1 = (0, \epsilon_1/2) \times \Gamma, \quad \Sigma_2 = (\epsilon_1/2, t_1 - \epsilon_1/2) \times \Gamma, \quad \Sigma_3 = (t_1 - \epsilon_1/2, t_1 + \epsilon_1/2) \times \Gamma.$$

We show that inequality (35) holds on each part.

For $(t, x) \in \Sigma_1$, we have $1 - \chi_{\epsilon_1}(t) = 0$. Consequently,

$$g(t,x)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda) = g^-(t,x,\lambda).$$

By the exponential decaying property of g^-, we obtain that

$$\|g(t,x)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_1)} \lesssim \sqrt{1}.$$

For $(t, x) \in \Sigma_2$, by the exponential decaying property for both g and g^- again, we obtain

$$\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_2)} \lesssim 1.$$

Finally, for $(t, x) \in \Sigma_3$, note that $t_1 - \frac{\epsilon_1}{2} = l(x_0, \xi_0) + \frac{\epsilon_1}{4} - \frac{\epsilon_1}{2} \geq \frac{3\epsilon_1}{4}$. We can apply Lemma 4.4 to the part $x \in V(x_1)$ and the exponential decaying property for both g and g^- to the remaining part to conclude that

$$\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_3)} \lesssim \sqrt{\lambda}$$

This completes the proof of (35) and hence the lemma.

5 Proof of Theorem 3.1

Proof of Theorem 3.1. For any $(x_0, \xi_0) \in S^*_-\Gamma$, let $(x_1, \xi_1) = \mathfrak{S}_c(x_0, \xi_0) = \mathcal{H}_c^{l(x_0,\xi_0)}(x_0, \xi_0)$ and $(\tilde{x}_1, \tilde{\xi}_1) = \mathfrak{S}_{\tilde{c}}(x_0, \xi_0) = \mathcal{H}_{\tilde{c}}^{\tilde{l}(x_0,\xi_0)}(x_0, \xi_0)$. We need to show that $(l(x_0, \xi_0), x_1, \xi_1) = (\tilde{l}(x_0, \xi_0), \tilde{x}_1, \tilde{\xi}_1)$ if $\|\Lambda_{\tilde{c}} - \Lambda_c\|$ is sufficiently small. We do this in the following steps.

Step 1. Let $t_1 = l(x_0, \xi_0) + \frac{\epsilon_1}{4}$ and $\tilde{t}_1 = \tilde{l}(x_0, \xi_0) + \frac{\epsilon_1}{4}$. Without loss of generality, we may assume that $t_1 \le \tilde{t}_1$. Let $V(x_1)$ be a neighborhood of x_1 in Γ which is parameterized by a smooth function $F : U(x_1) \to V(x_1)$ as before. We may assume that $\tilde{x}_1 \in V(x_1)$. Let $\tilde{x}_1 = F(\delta y)$. We construct the initial beam g, the reflected beam g^-, the boundary Dirichlet data f, the solution u to the wave equation with velocity field c and remanning term R as in the previous section. We similarly construct $\tilde{g}$, $\tilde{g}^-$, $\tilde{u}$ and $\tilde{R}$ to the system with velocity field $\tilde{c}$ and with boundary Dirichlet data $\tilde{f} = f$.

Step 2. Denote by $I(t_1, \epsilon_1/2)$ the interval $(t_1 - \epsilon_1/2, t_1 + \epsilon_1/2)$. Since $t_1 \le \tilde{t}_1$ and $l(x_0, \xi_0) \ge \epsilon_1$, we have $I(t_1, \epsilon_1/2) \subset (3\epsilon_1/4, t_1 + \epsilon_1/2)$ and $I(t_1, \epsilon_1/2) \subset (3\epsilon_1/4, \tilde{t}_1 + \epsilon_1/2)$. Then we can apply (31) and Lemma 4.5 to obtain

$$
\begin{aligned}
(\Lambda_{\tilde{c}} - \Lambda_c)f &= \frac{\partial u}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu} \\
&= \frac{\partial (g + g^-)}{\partial \nu} - \frac{\partial (\tilde{g} + \tilde{g}^-)}{\partial \nu} + \frac{\partial R}{\partial \nu} - \frac{\partial \tilde{R}}{\partial \nu} \\
&= 2i\lambda \cdot \left\{ \langle \xi_1, \nu(x_1) \rangle \cdot g - \langle \tilde{\xi}_1, \nu(\tilde{x}_1) \rangle \cdot \tilde{g} \right\} + O(\sqrt{\lambda})
\end{aligned}
$$

in $L^2(I(t_1, \epsilon_1/2) \times V(x_1))$.

It follows that

$$
\begin{aligned}
\big\langle (\Lambda_{\tilde{c}} - \Lambda_c)f, g \big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} = 2i\lambda \cdot \Big[&\big\langle \xi_1, \nu(x_1) \big\rangle \cdot \big\langle g, g \big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \\
&- \big\langle \tilde{\xi}_1, \nu(\tilde{x}_1) \big\rangle \cdot \big\langle \tilde{g}, g \big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \Big] + O(\sqrt{\lambda}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\langle (\Lambda_{\tilde{c}} - \Lambda_c)f, g \rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| &\le \|(\Lambda_{\tilde{c}} - \Lambda_c)f\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \cdot \|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \\
&\le \|(\Lambda_{\tilde{c}} - \Lambda_c)f\|_{L^2((0,T+\epsilon_1)\times \Gamma)} \cdot \|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \\
&\le \|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} \cdot \|f\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)} \\
&\quad \cdot \|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} \\
&\lesssim \lambda \cdot \|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)}.
\end{aligned}
$$

Thus the following inequality holds

$$
\begin{aligned}
&|\langle \xi_1, \nu(x_1) \rangle \cdot \langle g, g \rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| - |\langle \tilde{\xi}_1, \nu(\tilde{x}_1) \rangle \cdot \langle \tilde{g}, g \rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| \\
&\le \frac{C}{\sqrt{\lambda}} + C \cdot \|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} \qquad (36)
\end{aligned}
$$

for some constant $C > 0$.

Step 3. We now estimate the two terms on the left hand side of the inequality (36). First, by (8) and Lemma 4.3, we have

$$|\langle \xi_1, \nu(x_1)\rangle \cdot |\langle g, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| \approx 1. \tag{37}$$

We next estimate $\langle \tilde{g}, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}$. In the coordinate $x = F(y)$, by Lemma 4.2, we have

$$\langle \hat{\tilde{g}}, \hat{g}\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))} = \langle \hat{\tilde{g}}_*, \hat{g}_*\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))} + O(\frac{1}{\sqrt{\lambda}}). \tag{38}$$

Here we recall that $\hat{g}_* = a(t_1)e^{i\lambda\hat{\tau}_*}$ and $\hat{\tilde{g}}_* = \tilde{a}(t_1)e^{i\lambda\hat{\tilde{\tau}}_*}$ with

$$\begin{aligned}
\hat{\tau}_* &= \langle (-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1), (t-t_1, y)\rangle + (t-t_1, y)\hat{M}(t_1, 0)(t-t_1, y)^\dagger; \\
\hat{\tilde{\tau}}_* &= \langle (-1, \frac{\partial F}{\partial y}(\delta y)^\dagger \tilde{\xi}_1), (t-\tilde{t}_1, y-\delta y)\rangle + (t-\tilde{t}_1, y-\delta y)\hat{\tilde{M}}(t_1, 0)(t-\tilde{t}_1, y-\delta y)^\dagger.
\end{aligned}$$

By Lemma 8.1 in Appendix 8.3, we have

$$|\langle \hat{\tilde{g}}_*, \hat{g}_*\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))}| \lesssim e^{-c_0\lambda|\delta z|} \tag{39}$$

where c_0 is a positive constant depending only on $\|c\|_{C^3} + \|\tilde{c}\|_{C^3}$ and $|\delta z| = |t_1 - \tilde{t}_1|^2 + |\delta y|^2 + |\frac{\partial F}{\partial y}(\delta y)^\dagger \tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger \xi_1|^2$. It follows from (38) and (39) that

$$|\langle \tilde{g}, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| \lesssim e^{-c_0\lambda|\delta z|} + O(\frac{1}{\sqrt{\lambda}}). \tag{40}$$

Step 4. Combining (36), (37) and (40), we see that

$$e^{-c_0\lambda|\delta z|} \gtrsim C_1 - C_2\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} - C_3\frac{1}{\sqrt{\lambda}}$$

for some positive constants C_1, C_2 and C_3 which are independent of (x_0, ξ_0). By letting $\lambda \to \infty$, we conclude that $\delta z = 0$ if

$$\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} < \frac{C_1}{C_2}.$$

Set $\delta = \frac{C_1}{C_2}$. From $\delta z = 0$ it follows that $t_1 = \tilde{t}_1$, $\delta y = 0$, and $\frac{\partial F}{\partial y}(0)^\dagger\tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger\xi_1 = 0$. It remains to show that $\tilde{\xi}_1 = \xi_1$. Indeed, $\frac{\partial F}{\partial y}(0)^\dagger\tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger\xi_1 = 0$ implies that the tangential component of $\tilde{\xi}_1$ and ξ_1 are equal. Besides, $\|\xi_1\| = \|\tilde{\xi}_1\|$. These together with (8) yield that $\tilde{\xi}_1 = \xi_1$. This completes the proof of the theorem.

6 Geodesic X-ray transform with caustics

6.1 Local properties of the normal operator $\mathfrak{N}$

In this subsection, we present some results about the local properties of the normal operator $\mathfrak{N}$(see (16)).

From now on, we fix $x_* \in \Omega$. We first decompose $\mathfrak{N}$ locally into two parts based on the separation of singularities of its Schwartz kernel. Note that the map $\phi(x_*, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism in a neighborhood of the origin. In fact, we can check that $\frac{\partial \phi(x_*, \cdot)}{\partial \xi}(0) = c(x_*) \cdot Id$. Similar to the proof of existence of uniformly normal neighborhood in Riemannian manifold [17], we can find $\epsilon_2 > 0$ and a neighborhood of x_*, say $\tilde{U}(x_*) \subset \mathbb{R}^d$, such that

$$\phi(x, \cdot)|_{\|\xi\| < 2\epsilon_2} \text{ is a diffeomorphism for any } x \in \tilde{U}(x_*). \tag{41}$$

Let $\chi_{\epsilon_2} \in C_0^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $|t| < \epsilon_2$ and $\chi(t) = 0$ for $|t| > 2\epsilon_2$. We then define

$$\mathfrak{N}_1 f(x) = \int_{T_x^* \Omega} W(x, \xi) f(\phi(x, \xi)) \chi_{\epsilon_2}(\|\xi\|) \, d\sigma_x(\xi), \tag{42}$$

$$\mathfrak{N}_2 f(x) = \int_{T_x^* \Omega} W(x, \xi) f(\phi(x, \xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \, d\sigma_x(\xi). \tag{43}$$

Note that for any f supported in Ω, $f(\phi(x, \xi)) = 0$ for all $\|\xi\| > T$. Thus we have

$$\mathfrak{N}_2 f(x) = \int_{\xi \in T_x^* \Omega, \ \epsilon_2 < \|\xi\| < T} W(x, \xi) f(\phi(x, \xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \, d\sigma_x(\xi).$$

It is clear that $\mathfrak{N} f = \mathfrak{N}_1 f + \mathfrak{N}_2 f$. This gives the promised decomposition of $\mathfrak{N}$. We next study $\mathfrak{N}_1$ and $\mathfrak{N}_2$ separately.

Lemma 6.1. *$\mathfrak{N}_1$ is an elliptic ΨDO of order -1 from $C_0^\infty(\tilde{U}(x_*), \mathbb{R}^{2d})$ to $\mathcal{D}'(\tilde{U}(x_*), \mathbb{R}^{2d})$ with principle symbol*

$$\sigma_p(\mathfrak{N}_1)(x, \xi) = 2\pi \cdot \int_{S_x^* \Omega} \delta(\langle \xi, \theta \rangle) \Phi^\dagger(x, \theta) \cdot \Phi(x, \theta) \, d\sigma_x(\theta).$$

Proof. See [28] or [33].

We now proceed to study the operator $\mathfrak{N}_2$ whose property is determined by the Lagrangian map $\phi(x_*, \cdot)$. We shall study the operator germ $\mathfrak{N}_{2,\xi_*}$ for each $\xi \in T_{x_*}^* \mathbb{R}^d$. We first consider the case when ξ_* is not a caustic vector, i.e. ξ_* is a regular vector.

Lemma 6.2. *Let $\xi_* \in S_{x_*}^* \mathbb{R}^d$ be a regular vector, then there exists a neighborhood $U(x_*)$ of x_* and a neighborhood $B(x_*, \xi_*)$ of (x_*, ξ_*) such that for any $\chi \in C_0^\infty(B(x_*, \xi_*))$ the following operator*

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^* \Omega} W(x, \xi) f(\phi(x, \xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot \chi(x, \xi) \, d\sigma_x(\xi)$$

is a smoothing operator from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$.

Proof. Since $\xi_* \in S^*_{x_*}\mathbb{R}^d$ is regular, there exist a neighborhood $V(x_*)$ of x_* in $\mathbb{R}^d$ and a neighborhood $B(x_*,\xi_*)$ of (x_*,ξ_*) in $\mathbb{R}^{2d}$ of the form $B(x_*,\xi_*) = V(x_*) \times B_0(\xi_*)$ for some open set $B_0(\xi_*)$ in $\mathbb{R}^d$ such that the map $\phi(x,\cdot)$ is a diffeomorphism between $B_0(\xi_*)$ and its image for all $x \in V(x_*)$. We denote the inverse of the map $\phi(x,\cdot)$ by $\phi^{-1}(x,\cdot)$. By a change of coordinate $\xi = \phi^{-1}(x,y)$ and use some cut-off function, we can write $\mathfrak{N}_{2,\xi_*}$ in the following form

$$\mathfrak{N}_{2,\xi_*}f(x) = \int_\Omega K(x,y)f(y)\,dy, f \in \mathcal{E}'(\Omega,\mathbb{R}^{2d})$$

for some smooth function K in $\Omega \times \Omega$. The Lemma follows immediately.

We next consider the case when ξ_* is a fold vector. We have the following slightly modified result from [33].

Lemma 6.3. *Let ξ_* be a fold vector of the map $\phi(x_*,\cdot)$. Then there exists a small neighborhood $U(x_*)$ of x_* and a small neighborhood $B(x_*,\xi_*)$ of (x_*,ξ_*) in $\mathbb{R}^{2d}$ such that for any $\chi \in C_0^\infty(B(x_*,\xi_*))$, the operator $\mathfrak{N}_{2,\xi_*}: \mathcal{E}'(\Omega,\mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*),\mathbb{R}^{2d})$ defined by*

$$\mathfrak{N}_{2,\xi_*}f(x) = \int_{T_x^*\Omega} W(x,\xi)f(\phi(x,\xi))(1-\chi_{\epsilon_2}(\|\xi\|))\cdot\chi(x,\xi)\,d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega,\mathbb{R}^{2d}) \tag{44}$$

is an FIO of order $-\frac{d}{2}$ whose associated canonical relation is compactly supported in the following set

$$\Big\{(x,\xi,y,\eta); \qquad x \in U(x_*), y = \phi(x,\omega), (x,\omega) \in B(x_*,\xi_*),\ \det d_\omega\phi(x,\omega) = 0,$$
$$\xi = -\eta_i \frac{\partial \phi^i(x,\omega)}{\partial x},\ \eta \in \mathit{Coker}(d_\omega\phi(x,\omega)).\Big\} \tag{45}$$

Proof. We sketch a proof here and refer to [33] for detail. We first note that by the fold condition, there exists a small neighborhood $B_1(x_*,\xi_*)$ of (x_*,ξ_*) in $\mathbb{R}^{2d}$ such that ξ_* is the only singular vector of the map $\phi(x_*,\cdot)$ along the ray $\{t\xi_* : t \in \mathbb{R}\}$ in $B_1(x_*,\xi_*)$. Define

$$\begin{aligned} S &= \{(x,w) : \det d_\omega\phi(x,\omega) = 0\} \subset \mathbb{R}^{2d}, \\ \Sigma &= \{(x,y) : y = \phi(x,\omega), (x,\omega) \in S\} \subset \mathbb{R}^{2d}. \end{aligned}$$

By shrinking $B_1(x_*,\xi_*)$ if necessary, we can show that $S\bigcap B_1(x_*,\xi_*)$ is a smooth $(2d-1)$-dimensional manifold in $\mathbb{R}^{2d}$, and ϕ is a diffeomorphism between $S\bigcap B_1(x_*,\xi_*)$ and its image. Denote

$$S_1 = S\bigcap B_1(x_*,\xi_*), \quad \Sigma_1 = \phi(S_1).$$

Note that Σ_1 is a smooth $(2d-1)$-dimensional manifold in a neighborhood of (x_*,y_*) in $\Omega\times\Omega$. Let π_2 be the projection from $\Omega\times\Omega$ to its second component. By the fold-condition for ξ_* and the fact that the matrix $d_{x,\xi}\phi(x_*,\xi_*)$ is surjective, we can show that $d\pi_2 : T_{(x_*,\xi_*)}\Sigma_1 \to T_{y_*}\mathbb{R}^d$ is surjective. Thus there exists a neighborhood $V_1(y_*)$ of y_* in $\mathbb{R}^d$ such that $V_1(y_*) \subset \pi_2(\Sigma_1)$. We remark that the surjectivity of the linear map $d\pi_2$ implies that the conormal bundle of Σ_1 belongs to $(T^*\mathbb{R}^d \setminus 0) \times (T^*\mathbb{R}^d \setminus 0)$, where 0 stands for the zero section of the normal bundle $T^*\mathbb{R}^d$.

Let $U_1(x_*)$ be a small neighborhood of x_* such that $U_1(x_*) \subset \pi(S_1)$ and $\tilde\chi \in C_0^\infty(B_1(x_*,\xi_*))$.

Consider the Schwartz kernel of operator $\tilde{\mathfrak{N}}_{2,\xi_*} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U_1(x_*), \mathbb{R}^{2d})$ defined by

$$\tilde{\mathfrak{N}}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot \tilde{\chi}(x,\xi)\, d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d}).$$

We can show that it has conormal singularity supported in the set Σ_1. Moreover, the conormal bundle $\mathcal{N}^*\Sigma_1$ is given by

$$\mathcal{N}^*\Sigma_1 = \Big\{(x,\xi,y,\eta); \qquad x \in U(x_*), y = \phi(x,\omega), (x,\omega) \in B_1(x_*,\xi_*), \\ \xi = -\eta_i \frac{\partial \phi^i(x,\omega)}{\partial x}, \eta \in \mathrm{Coker}\,(d_\omega \phi(x,\omega)),\ \det d_\omega \phi(x,\omega) = 0.\Big\}$$

By analyzing the singularity of the Jacobian determinant of $d_\omega \phi(x,\omega))$, we can show that the Schwartz kernel of $\tilde{\mathfrak{N}}_{2,\xi_*}$ belongs to the conormal class $I^{-\frac{d}{2}}(\Omega \times \Omega, \Sigma_1, \mathcal{M}_{2d\times 2d})$, where $\mathcal{M}_{2d\times 2d}$ denotes the vector bundle of matrices from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$ over Ω. Especially, when the domain of $\tilde{\mathfrak{N}}_{2,\xi_*}$ is restricted to distribution sections supported in $V_1(y_*)$, the operator $\tilde{\mathfrak{N}}_{2,\xi_*}$ is a FIO of order $-\frac{d}{2}$ from $\mathcal{E}'(V_1(y_*), \mathbb{R}^{2d})$ to $\mathcal{D}'(U_1(x_*), \mathbb{R}^{2d})$.

The above define $\tilde{\mathfrak{N}}_{2,\xi_*}$ with $\tilde{\chi} \in C_0^\infty(B_1(x_*,\xi_*))$ requires the domain to be $\mathcal{E}'(V_1(y_*), \mathbb{R}^{2d})$, we now show that this condition can be relaxed by further decreasing $B_1(x_*,\xi_*)$. Indeed, let $U(x_*)$ be a neighborhood of x_* such that $U(x_*) \Subset U_1(x_*)$ and $B(x_*,\xi_*)$ be a neighborhood of (x_*,ξ_*) such that $B(x_*,\xi_*) \Subset B_1(x_*,\xi_*)$. By choosing $U(x_*)$ and $B(x_*,\xi_*)$ to be sufficiently small, we can assume that the set $V(y_*) = \{\ \phi(x,\xi) : x \in U(x_*), \xi \in B(x_*,\xi_*)\}$ is compactly supported in $V_1(y_*)$. We then choose $\chi_V \in C_0^\infty(V_1(x_*))$ such that $\chi_V(x) = 1$ for $x \in V(x_*)$. One can check that for any $\chi \in C_0^\infty(B(x_*,\xi_*))$, the operator $\mathfrak{N}_{2,\xi_*}$ defined by

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot \chi(x,\xi)\, d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d})$$

satisfies $\mathfrak{N}_{2,\xi_*} f(x) = \mathfrak{N}_{2,\xi_*}(\chi_V \cdot f)(x)$ for all $x \in U(x_*)$. Thus $\mathfrak{N}_{2,\xi_*}$ is well-defined from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*), \mathbb{R}^{2d})$, and is a FIO of order $-\frac{d}{2}$ with canonical relation compactly supported in the set (45). This completes the proof of the lemma.

6.2 Singularities of the map $\phi(x,\cdot)$

In this subsection, we present some properties about the map $\phi(x,\cdot)$ which is equivalent to the exponential map in Riemannian manifold.

By the classification result for Lagrangian maps (see [3] and [4] for detail), there are only a finite number of stable and simple singular Lagrangian map germs in dimensions between three and five and they are generic. In three dimensions, there are four types: fold, cusp, swallow-tail and D4. The others are unstable and can be removed by using arbitrarily small perturbations.

We define

$$
\begin{aligned}
\mathcal{K}(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ is singular}\};\\
\mathcal{K}_1(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity of fold type}\};\\
\mathcal{K}_2(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity of cusp type}\};\\
\mathcal{K}_3(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has simple and stable singularities of types}\\
&\qquad \text{other than fold and cusp}\};\\
\mathcal{K}_4(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity which are either not simple or stable}\}.
\end{aligned}
$$

It is clear that $\mathcal{K}(x) = \bigcup_{j=1}^4 \mathcal{K}_j(x)$. Denote

$$
\begin{aligned}
&\mathcal{S}(x) = \{\xi \in S_x^*\mathbb{R}^d : r\xi \in \mathcal{K} \text{ for some } \ r \in \mathbb{R}\},\\
&\mathcal{S}_j(x) = \{\xi \in S_x^*\mathbb{R}^d : r\xi \in \mathcal{K}_j \text{ for some } \ r \in \mathbb{R}\}, \ \ j = 1,2,3,4.
\end{aligned}
$$

We say that the map $\phi(x,\cdot)$ is in a general position(or generic) if the map germ $\phi(x,\cdot)$ is simple and stable at all caustic vectors in $\mathcal{K}(x)$, i.e. $\mathcal{K}_4(x) = \emptyset$. It is possible that the map $\phi(x,\cdot)$ can be brought to a general position by adding an arbitrarily small perturbation to the velocity field c. By the classification result of Lagrangian maps, see for instance [3], the following result holds for the set $\mathcal{K}(x)$.

Propsition 1. *Assume that the map $\phi(x,\cdot)$ is in a general position, then the sets $\mathcal{K}_1(x)$ and $\mathcal{K}_2(x)$ are smooth manifolds of dimensions $d-1$ and $d-2$, respectively. The set $\mathcal{K}_3(x)$ is a union of smooth manifolds of dimensions not greater than $d-3$. Especially, for $d=3$, the sets $\mathcal{K}_1(x)$, $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ consists of smooth surfaces, smooth curves and isolated points, respectively.*

In the case when the map $\phi(x,\cdot)$ is not in a general position, it is known that $\mathcal{K}_1(x)\bigcup\mathcal{K}_2(x)\bigcup\mathcal{K}_3(x)$ is open and dense in $\mathcal{K}(x)$.

Note that $\mathcal{S}_j(x)$ are the images of $\mathcal{K}_j(x)$ under the map β which sends $\xi \in T_x^*\mathbb{R}^d$ to $\frac{\xi}{\|\xi\|} \in S_x^*\mathbb{R}^d$ for $\xi \neq 0$. We conclude that the following result holds.

Lemma 6.4. *Assume that the map $\phi(x,\cdot)$ is in general position, then the sets $\mathcal{S}_2(x)$ and $\mathcal{S}_3(x)$ are of finite $d-2$ and $d-3$ dimensional Hausdorff measures, respectively. Especially, for $d=3$, the set $\mathcal{S}_2(x)$ is a curve (not necessarily smooth) of finite length in $S_x^*\mathbb{R}^3$ and $\mathcal{S}_3(x)$ consists of a finite number of points.*

6.3 Discussions on the concept of Fold-regular

In this subsection, we discuss the concept "fold-regular". We show that for a general velocity field in Ω whose induced metric is not simple, a given point in Ω is fold-regular under some natural assumptions.

We begin with the concept "fold-regular vector". It is still an open problem to find a complete characterization for it, i.e. what are the necessary and sufficient conditions for the map germ $\phi(x_*,\cdot)$ at ξ_* for ξ_* to be fold-regular. We have the following partial answer in the form of remarks.

Remark 6.1. *In dimension $d=2$, the set of fold-regular vectors is generally empty. Indeed, for a fold vector ξ_*, the operator germ $\mathfrak{N}_{2,\xi_*}$ is a FIO of order -1, and hence the best estimate is that it is bounded from $L^2(\Omega_{\epsilon_0},\mathbb{R}^{2d})$ to $H^1(U(x_*),\mathbb{R}^{2d})$ for some neighborhood $U(x_*)$ of x_*.*

Remark 6.2. *In dimension $d \geq 3$, a sufficient condition for a fold vector ξ_* to be fold-regular is that the following condition is satisfied*

$$d^2_\xi \phi(x_*, \xi_*)(N_{x_*}(\xi_*) \setminus 0 \times \cdot)|_{T_{\xi_*} S(x_*)} \quad \textit{is of full rank.} \tag{46}$$

where $N_{x_*}(\xi_*)$ denotes the kernel of $d_\xi \phi(x_*, \xi_*)$ and $S(x_*)$ the set of all vectors $\xi \in T^*_{x_*}\mathbb{R}^d$ such that $\det d_\xi \phi(x_*, \xi) = 0$. Indeed, in that case, it is shown in [33] that the canonical relation associated with the operator germ $\mathfrak{N}_{2,\xi_*}$ is locally a canonical graph and hence $\mathfrak{N}_{2,\xi_*}$ is bounded from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^{\frac{d}{2}}(U(x_*), \mathbb{R}^{2d})$ for some neighborhood $U(x_*)$ of x_*. Note that for $d \geq 3$, $H^{\frac{d}{2}}(U(x_*), \mathbb{R}^{2d})$ is compactly embedded in $H^1(U(x_*), \mathbb{R}^{2d})$, so $\mathfrak{N}_{2,\xi_*}$ is compact from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^1(U(x_*), \mathbb{R}^{2d})$ and we can conclude that ξ_* is fold-regular.

The set of fold-regular vectors $\mathcal{Z}_1(x_*)$ contains more elements than those which satisfy the graph condition (46). In fact, let $\mathcal{C} \subset T^*\Omega \times T^*\Omega$ be the canonical relation associated with the operator germ $\mathfrak{N}_{2,\xi_*}$ defined in Lemma 6.3. We have shown that $\mathcal{C}$ is homogeneous and $\mathcal{C} \subset (T^*\Omega \backslash 0) \times (T^*\Omega \backslash 0)$. By the main result in [12], $\mathfrak{N}_{2,\xi_*}$ is bounded from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^{\frac{d}{2}-\frac{1}{3}}(U(x_*), \mathbb{R}^{2d})$ for some neighborhood $U(x_*)$ of x_*, if the only singularity of the projection of $\mathcal{C}$ to its first or second component at the point associated with (x_*, ξ_*) is fold or cusp. Since $H^{\frac{d}{2}-\frac{1}{3}}(U(x_*), \mathbb{R}^{2d})$ is compactly embedded in $H^1(U(x_*), \mathbb{R}^{2d})$, we see that $\mathfrak{N}_{2,\xi_*}$ is compact from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^1(U(x_*), \mathbb{R}^{2d})$ and hence ξ_* is fold-regular.

We now consider the concept "fold-regular point". We denote

$$\mathcal{Z}_1(x_*) = \{\xi \in S^*_{x_*}\mathbb{R}^d : \ \forall\, r \in \mathbb{R}, r\xi \text{ is either regular or fold-regular for the map } \phi(x_*, \cdot)\}.$$

It is clear that

$$\mathcal{Z}_1(x_*) \subset \mathcal{Z}(x_*) =: S^*_{x_*}\mathbb{R}^d \setminus \overline{(\mathcal{S}_2(x_*) \bigcup \mathcal{S}_3(x_*)) \bigcup \mathcal{S}_4(x_*)}.$$

We remark that $\mathcal{Z}(x_*)$ characterizes the set of geodesics that pass through x_* and along which the map $\phi(x_*, \cdot)$ only has singularities of fold type. By Morse's index theorem (a fold vector for the map $\phi(x, \cdot)$ corresponds to a fold conjugate vector for the exponential map $\exp_x(\cdot)$), for each $\xi \in \mathcal{S}(x_*)$, there are at most finitely many fold vectors along the geodesic $\pi \circ \mathcal{H}^t(x_*, \xi)$. Using the definition of "fold-regular", we can conclude that $\mathcal{Z}_1(x_*)$ is open in $S^*_{x_*}\mathbb{R}^d$.

Recall that x_* is fold-regular if there exists a compact subset $\mathcal{Z}_2(x_*) \subset \mathcal{Z}_1(x_*)$ such that the following completeness condition is satisfied

$$\forall\, \xi \in S^*_{x_*}\mathbb{R}^d, \exists\, \theta \in \mathcal{Z}_2(x_*), \quad \text{such that } \theta \perp \xi.$$

Remark 6.3. *A sufficient condition for the completeness of $\mathcal{Z}_1(x_*)$ is that $\mathcal{Z}_1(x_*)$ contains a set*

$$\theta^\perp = \{\xi \in S^*_{x_*}\mathbb{R}^d, \xi \perp \theta\}$$

for $\theta \in S^*_{x_*}\mathbb{R}^d$.

Remark 6.4. *If the completeness condition fails for $\mathcal{Z}_1(x_*)$, then there exists $\theta \in S^*_{x_*}\mathbb{R}^d$ such that*

$$\theta^\perp \subset \overline{\mathcal{S}_2(x_*) \bigcup \mathcal{S}_3(x_*) \bigcup \mathcal{S}_4(x_*)}. \tag{47}$$

Assume that the map $\phi(x_*, \cdot)$ is in a general position. By Lemma 6.4, the set $\overline{\mathcal{S}_2(x_*) \bigcup \mathcal{S}_3(x_*) \bigcup \mathcal{S}_4(x_*)}$

is of finite $d-2$-dimensional Hausdorff measure, so is the set $\theta^\perp$ for each $\theta \in S^*_{x_*}\mathbb{R}^d$. Thus, we conclude that there exists at most a finite number of θ such that (47) holds.

6.4 Proof of Theorem 3.2

We prove Theorem 3.2 in this subsection. The proof can be divided into two major stages: in the first stage, we present some preliminaries and construct a cut-off function $\alpha \in C_0^\infty(S^*_-\Gamma)$ which selects a complete set of geodesics with only fold-regular caustics, see Lemma 6.5; in the second stage, we study the normal operator $\mathfrak{N}_\alpha = \mathfrak{I}_\alpha^\dagger \mathfrak{I}_\alpha$, see Lemma 6.6 and 6.7. Theorem 3.2 is then a direct consequence of Lemma 6.6 and 6.7.

We now present some preliminaries that are necessary for the construction of α. Let x_* be a fold-regular point with the compact subset $\mathcal{Z}_2(x_*) \subset S^*_{x_*}\mathbb{R}^d$ in Definition 3.4. Denote $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2 = \{r\xi; \xi \in \mathcal{Z}_2(x_*), r \in \mathbb{R} \quad \text{and} \ \epsilon_2 \le |r| \le T\}$. For each $\xi_* \in \mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2$, by Lemma 6.2 and Lemma 6.3, there exist a neighborhood $U(x_*, \xi_*)$ of x_* and a neighborhood $B(x_*, \xi_*)$ of (x_*, ξ_*) such that for any $\chi \in C_0^\infty(B(x_*, \xi_*))$ the following operator

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1-\chi_{\epsilon_2}(\|\xi\|)) \cdot \chi(x,\xi)\, d\sigma_x(\xi)$$

is compact from $H^s(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^{s+1}(U(x_*, \xi_*), \mathbb{R}^{2d})$. Let $B_0(x_*, \xi_*)$ be another neighborhood of (x_*, ξ_*) in $\mathbb{R}^{2d}$ such that $B_0(x_*, \xi_*) \Subset B(x_*, \xi_*)$. Since $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2$ is compact, there exists a finite number of ξ_*'s in $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2$, say ξ_1, ξ_2,..., ξ_M, such that

$$\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2 \subset \bigcup_{j=1}^M B_0(x_*, \xi_j).$$

We can then find smooth functions χ_1, χ_2,...,χ_M with $\mathrm{supp}\chi_j \subset B(x_*, \xi_j)$ for each j such that

$$\sum_{j=1}^M \chi_j(x,\xi) = 1 \quad \text{for all} \quad (x,\xi) \in \bigcup_{j=1}^M B_0(x_*, \xi_j).$$

Denote by $\mathcal{A}_0$ be the greatest connected open symmetric subset in $\bigcup_{j=1}^M B_0(x_*, \xi_j)$ which contains $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2$. Here and after, we say that a set $\mathcal{B}$ in $\mathbb{R}^{2d}$ is symmetric if $(x,\xi) \in \mathcal{B}$ implies that $(x, -\xi) \in \mathcal{B}$. Define

$$\mathcal{A}_\epsilon = \{(x,\xi) \in \mathbb{R}^{2d} : |x - x_*| \le \epsilon, \ \epsilon_2 \le \|\xi\| \le T\}$$

for each $\epsilon > 0$. It is clear that $\mathcal{A}_\epsilon$ is compact in $\mathbb{R}^{2d}$, so is the set $\mathcal{A}_\epsilon \backslash \mathcal{A}_0$.

Lemma 6.5. *There exist $\epsilon_3 > 0$ and $\alpha \in C_0^\infty(S^*_-\Gamma)$ such that the following two conditions are satisfied:*

$$\begin{aligned} \alpha(x_0,\xi_0) &= 1 \quad \textit{for all} \ (x_0,\xi_0) \in \tau \circ \beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*)), && (48)\\ \alpha(x_0,\xi_0) &= 0 \quad \textit{for all} \ (x_0,\xi_0) \in \tau \circ \beta(\mathcal{A}_{\epsilon_3} \backslash \mathcal{A}_0). && (49) \end{aligned}$$

Proof. Note that both β and τ are continuous. Since $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*)$ and $\mathcal{A}_\epsilon \backslash \mathcal{A}_0$ are compact,

so are the sets $\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))$ and $\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)$. We claim that there exists $\epsilon_3>0$ such that

$$\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)=\emptyset$$

for all $\epsilon\le\epsilon_3$. Indeed, assume the contrary, then

$$\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)\neq\emptyset$$

for all $\epsilon>0$. Note that the collection of compact subsets $\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)$ is decreasing with respect to ϵ, so it satisfies the finite intersection property and we can thus conclude that

$$\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap_{\epsilon>0}\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)\neq\emptyset.$$

But on the other hand, we can check that

$$\begin{aligned}\bigcap_{\epsilon>0}\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0) &= \tau((\mathcal{A}_\epsilon\backslash\mathcal{A}_0)\bigcap S^*_{x_*}\mathbb{R}^d)\\ \tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*)) &= \tau(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*)\bigcap S^*_{x_*}\mathbb{R}^d).\end{aligned}$$

Using the fact that τ is injective on $S^*_{x_*}\mathbb{R}^d$ and $\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2\subset\mathcal{A}_0$, we obtain

$$\tau((\mathcal{A}_\epsilon\backslash\mathcal{A}_0)\bigcap S^*_{x_*}\mathbb{R}^d)\bigcap\tau(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*)\bigcap S^*_{x_*}\mathbb{R}^d)=\emptyset.$$

Thus,

$$\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap_{\epsilon>0}\tau\circ\beta(\mathcal{A}_\epsilon\backslash\mathcal{A}_0)=\emptyset.$$

This contradiction completes the proof of our claim.

Now, we have

$$\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))\bigcap\tau\circ\beta(\mathcal{A}_{\epsilon_3}\backslash\mathcal{A}_0)=\emptyset.$$

By decreasing ϵ_3 if necessary, we may assume that

$$\{x:|x-x_*|\le\epsilon_3\}\subset\pi(\mathcal{A}_0).$$

Since both the sets $\tau\circ\beta(\mathcal{C}_{\epsilon_2,T}\mathcal{Z}_2(x_*))$ and $\tau\circ\beta(\mathcal{A}_{\epsilon_3}\backslash\mathcal{A}_0)$ are compact in $S^*_-\Gamma$, we can find $\alpha\in C_0^\infty(S^*_-\Gamma)$ as desired. This concludes the proof of the lemma.

The construction of α above completes the first stage of the proof of Theorem 3.2, we are now at the second stage. We define the truncated geodesic X-ray transform $\mathfrak{I}_\alpha f$ as in (19) or (20). By replacing the weight Φ with the new one $\alpha^\sharp\cdot\Phi$, we obtain $\mathfrak{N}_\alpha$, $\mathfrak{N}_{1,\alpha}$ and $\mathfrak{N}_{2,\alpha}$ from the corresponding formulas of $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$. It is clear that Lemma 6.2, 6.3 still hold with the new weight.

Lemma 6.6. *There exist a neighborhood $U(x_*)$ of x_* and a smoothing operator $\mathfrak{R}$ from $\mathcal{E}'(\Omega,\mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)},\mathbb{R}^{2d})$, such that for for any $s\in\mathbb{R}$ and any neighborhood $U_0(x_*)$ of x_* with $U_0(x_*)\Subset U(x_*)$, the following estimate holds*

$$\|f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}\lesssim\|\mathfrak{N}_{1,\alpha}f\|_{H^{s+1}(U(x_*),\mathbb{R}^{2d})}+\|\mathfrak{R}f\|_{H^s(\Omega,\mathbb{R}^{2d})}. \tag{50}$$

Proof. We first show that $\mathfrak{N}_{1,\alpha}$ is an elliptic ΨDO. Indeed, as in Lemma 6.1, $\mathfrak{N}_{1,\alpha}$ is a ΨDO of order -1 from $C_0^\infty(\tilde{U}(x_*),\mathbb{R}^{2d})$ to $\mathcal{D}'(\tilde{U}(x_*),\mathbb{R}^{2d})$ with principle symbol

$$\sigma_p(\mathfrak{N}_1)(x,\xi) = 2\pi \cdot \int_{S_x^*\Omega} \delta(\langle \xi,\theta\rangle)|\alpha^\sharp(x_*,\theta)|^2 \Phi^\dagger(x,\theta)\cdot\Phi(x,\theta)\, d\sigma_x(\theta).$$

By the construction of α, for any $\xi \in S_{x_*}^*\mathbb{R}^d$, we have $\alpha^\sharp(x_*,\theta)=1$ for some $\theta \in S_{x_*}^*\mathbb{R}^d$ with $\theta \perp \xi$. Thus

$$\sigma_p(\mathfrak{N}_{1,\alpha})(x_*,\xi) = 2\pi \cdot \int_{\theta\in S_{x_*}^*\mathbb{R}^d,\,\theta\perp\xi} |\alpha^\sharp(x_*,\theta)|^2 \Phi^\dagger(x_*,\theta)\cdot\Phi(x_*,\theta)\, d\sigma_{x_*}(\theta) > 0$$

in the sense of symmetric positive definite matrix. By continuity, we can find a neighborhood $U(x_*) \subset \tilde{U}(x_*)$ of x_* such that $\sigma_p(\mathfrak{N}_{1,\alpha})(x,\xi) > 0$ for all $x \in U(x_*)$ and $\xi \in S_x^*\mathbb{R}^d$. Thus we can conclude that $\mathfrak{N}_{1,\alpha}$ is an elliptic ΨDO of order -1 from $C_0^\infty(U(x_*),\mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*),\mathbb{R}^{2d})$.

Now, let $\mathfrak{B}$ be the pseudo-inverse of $\mathfrak{N}_{1,\alpha}$ restricted to $U(x_*)$. Then $\mathfrak{B}$ is ΨDO of order 1 and there is a smoothing operator $\mathfrak{R}_1 : \mathcal{E}'(U(x_*),\mathbb{R}^{2d}) \to C^\infty(U(x_*),\mathbb{R}^{2d})$ such that

$$g = \mathfrak{B}\circ\mathfrak{N}_{1,\alpha} g + \mathfrak{R}_1 g \tag{51}$$

for all g supported in $U(x_*)$.

Next, let $U_0(x_*)$ be any given neighborhood of x_* with $U_0(x_*) \Subset U(x_*)$. For later convenience, we write $U_3(x_*)$ for $U(x_*)$. Then there exist two neighborhoods of x_*, say $U_1(x_*)$ and $U_2(x_*)$ such that $U_0(x_1) \Subset U_1(x_*) \Subset U_2(x_*) \Subset U_3(x_*)$. We choose three smooth cut-off functions χ_0, χ_1 and χ_2 such that $\operatorname{supp}\chi_j \subset U_{j+1}$ and $\chi_j|_{U_j} = 1$ for $j = 0,1,2$. Then both $\chi_0\mathfrak{B}(1-\chi_1)$ and $\chi_1\mathfrak{N}_{1,\alpha}(1-\chi_2)$ are smoothing operators.

Note that $\chi_2 f$ is compact supported in $U_3(x_*)$, so we have by (51) that

$$\chi_2 f = \mathfrak{B}\circ\mathfrak{N}_{1,\alpha}(\chi_2 f) + \mathfrak{R}_1(\chi_2 f).$$

Thus,

$$\begin{aligned}
\chi_0 f &= \chi_0\cdot\chi_2 f = \chi_0\cdot\mathfrak{B}\circ\mathfrak{N}_{1,\alpha}\chi_2 f + \chi_0\mathfrak{R}_1(\chi_2 f)\\
&= \chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha}\chi_2\cdot f + \big(\chi_0\mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0\mathfrak{R}_1\chi_2\big)f\\
&= \chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha} f + \big(\chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha}(1-\chi_2) + \chi_0\mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0\mathfrak{R}_1\chi_2\big)f\\
&= \chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha} f + \mathfrak{R} f
\end{aligned}$$

where $\mathfrak{R} = \chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha}(1-\chi_2) + \chi_0\mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0\mathfrak{R}_1\chi_2$. We can check that $\mathfrak{R}$ is a smoothing operator from $\mathcal{E}'(\Omega,\mathbb{R}^{2d})$ to $C^\infty(\overline{U(x_*)},\mathbb{R}^{2d})$.

Finally, we conclude that

$$\begin{aligned}
\|f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} &\lesssim \|\chi_0\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}\\
&\lesssim \|\mathfrak{B}\chi_1\mathfrak{N}_{1,\alpha} f\|_{H^s(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}\\
&\lesssim \|\chi_1\cdot\mathfrak{N}_{1,\alpha} f\|_{H^{s+1}(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}\\
&\lesssim \|\mathfrak{N}_{1,\alpha} f\|_{H^{s+1}(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}.
\end{aligned}$$

This completes the proof of the Lemma.

We now study the operator $\mathfrak{N}_{2,\alpha}$.

Lemma 6.7. *There exists a small neighborhood of x_*, say $U(x_*)$, such that the following decomposition holds for the operator $\mathfrak{N}_{2,\alpha} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*), \mathbb{R}^{2d})$*

$$\mathfrak{N}_{2,\alpha} = \sum_{j=1}^{M} \mathfrak{N}_{2,j} \tag{52}$$

where each $\mathfrak{N}_{2,j}$ is compact from $H^s(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^{s+1}(U(x_*), \mathbb{R}^{2d})$.

Proof. Recall that $\mathfrak{N}_{2,\alpha}$ has the following representation

$$\mathfrak{N}_{2,\alpha} f(x) = \int_{T_x^*\Omega} W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot d\sigma_x(\xi),$$

where

$$\begin{aligned} W_\alpha(x,\xi) \quad = \quad & \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,\xi)|^2 \Phi^\dagger \circ \beta(x,\xi) \cdot \Phi \circ \beta \circ \mathcal{H}(x,\xi) \\ & + \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,-\xi)|^2 \Phi^\dagger \circ \beta(x,-\xi) \cdot \Phi \circ \beta \circ \mathcal{H}^{-1}(x,-\xi). \end{aligned}$$

By (49) and the fact that $\mathcal{A}_0$ is symmetric, we see that supp $W_\alpha \subset \mathcal{A}_0$ for all x with $|x - x_*| \le \epsilon_3$.

Now, let χ_j's be as in the first stage. Define $\mathfrak{N}_{2,j} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*, \xi_j), \mathbb{R}^{2d})$ by

$$\mathfrak{N}_{2,j} f(x) = \int_{T_x^*\Omega} W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot \chi_j(x,\xi)\, d\sigma_x(\xi).$$

Let $U(x_*) = \bigcap_{j=1}^{M} (U(x_*, \xi_j)) \bigcap \{x : |x - x_*| < \epsilon_3\}$. Then $U(x_*)$ is a neighborhood of x_* and each $\mathfrak{N}_{2,j}$ is compact from $H^s(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ into $H^{s+1}(U(x_*), \mathbb{R}^{2d})$.

We claim that $\mathfrak{N}_{2,\alpha} = \sum_{j=1}^{M} \mathfrak{N}_{2,j}$ when both sides are viewed as operators from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*), \mathbb{R}^{2d})$. Indeed, for any $f \in C_0^\infty(\Omega, \mathbb{R}^{2d})$, since $\sum_{j=1}^{M} \chi_j = 1$ on $\mathcal{A}_0$ and supp $W_\alpha \subset \mathcal{A}_0$, we have

$$W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) = W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_{\epsilon_2}(\|\xi\|)) \cdot \big(\sum_{j=1}^{M} \chi_j(x,\xi)\big)$$

for all $x \in U(x_*)$. Thus $\mathfrak{N}_{2,\alpha} f = \sum_{j=1}^{M} \mathfrak{N}_{2,j} f$ and the claim follows. This completes the proof of the lemma.

Finally, note that $\mathfrak{N}_\alpha = \mathfrak{N}_{1,\alpha} + \mathfrak{N}_{2,\alpha}$. Theorem 3.2 follows from Lemma 6.6 and 6.7.

7 Sensitivity analysis of recovering the velocity field from the DDtN map

In this section, we prove Theorem 3.3 on the sensitivity of the inverse problem of recovering the velocity field from the DDtN map. We first present a lemma which is a direct consequence of Theorem 3.1 and Lemma 3.3.

Lemma 7.1. *Let c and $\tilde{c}$ be two velocity field in $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$, and let f be as in (14). Then there exists $\delta > 0$ such that if $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$, then*

$$\|\mathfrak{I}f\|_{L^\infty(S_-^*\Gamma, \mathbb{R}^{2d})} \leq C\|f\|^2_{C^1(\Omega,\mathbb{R}^{2d})} \tag{53}$$

for constant $C > 0$ depending M_0.

Proof of Theorem 3.3. The proof is divided into the following six steps.

Step 1. Since c is fold-regular, for each $x \in \overline{\Omega}_{\epsilon_0}$, by Theorem 3.2, there exist a neighborhood $U(x)$ of x, a smooth cut-off function $\alpha \in C_0^\infty(S_-^*\Gamma)$ and a smoothing operator $\mathfrak{R}$ such that for any $U_0(x) \Subset U(x)$ the following estimate holds

$$\|f\|_{L^2(U_0(x),\mathbb{R}^{2d})} \lesssim \|\mathfrak{N}_\alpha f\|_{H^1(U(x),\mathbb{R}^{2d})} + \|\mathfrak{N}_{2,\alpha} f\|_{H^1(\Omega,\mathbb{R}^{2d})} + \|\mathfrak{R}f\|_{H^1(\Omega,\mathbb{R}^{2d})} \tag{54}$$

for all $f \in L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$. Moreover, both $\mathfrak{N}_{2,\alpha}$ and $\mathfrak{R}$ are compact from $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ to $H^1(U(x), \mathbb{R}^{2d})$.

We now fix a neighborhood $U_0(x) \Subset U(x)$ of x for each x. Note that $\overline{\Omega}_{\epsilon_0}$ is compact, there exists a finite number of points, say x_1, x_2, ... x_M such that $\overline{\Omega}_{\epsilon_0} \subset \bigcup_{j=1}^M U_0(x_j)$. Let $\mathfrak{N}_{\alpha_j}$ be the operator associated with each point x_j.

Step 2. Denote by H the Hilbert space $\prod_{j=1}^M H^1(U(x_j), \mathbb{R}^{2d})$. We consider the following three operators

$$\begin{aligned} Tf &= (\mathfrak{N}_{\alpha_1} f, \mathfrak{N}_{\alpha_2} f, ..., \mathfrak{N}_{\alpha_M} f), \\ T_1 f &= (\mathfrak{N}_{2,\alpha_1} f, \mathfrak{N}_{2,\alpha_2} f, ..., \mathfrak{N}_{2,\alpha_M} f), \\ T_2 f &= (\mathfrak{R}_{\alpha_1} f, \mathfrak{R}_{\alpha_2} f, ..., \mathfrak{R}_{\alpha_M} f). \end{aligned}$$

It is clear that all three operators are bounded from $L^2(\Omega_{\epsilon_0})$ to H. Moreover, T_1 and T_2 are also compact and the following estimate holds

$$\|f\|_{L^2(\Omega,\mathbb{R}^{2d})} \lesssim \|Tf\|_H + \|T_1 f\|_H + \|T_2 f\|_H. \tag{55}$$

Step 3. Let $\mathfrak{L}_0 \subset L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ be the kernel of T. We claim that $\mathfrak{L}_0 \subset L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ is of finite dimension. We prove by contradiction. Assume the contrary, then there exists an infinity number of orthogonal vectors in $\mathfrak{L}_0 \subset L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$, say, e_1, e_2, ..., such that $\|e_j\|_{L^2(\Omega_{\epsilon_0},\mathbb{R}^{2d})} = 1$ and $Te_j = 0$ for all $j \in \mathbb{N}$. Since the sequence $\{e_j\}_{j=1}^\infty$ is bounded in $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$ and the operators T_1 and T_2 are compact, we can find a subsequence, still denoted by $\{e_j\}_{j=1}^\infty$, such that both the sequences $\{T_1 e_j\}_{j=1}^\infty$ and $\{T_2 e_j\}_{j=1}^\infty$ are Cauchy in H. By applying Inequality (55) to the vectors $e_i - e_j$ and recall that $T(e_i - e_j) = 0$, we conclude that the sequence $\{e_j\}_{j=1}^\infty$ is also Cauchy in $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$. This contradicts to the fact that $\|e_i - e_j\|_{L^2(\Omega_{\epsilon_0},\mathbb{R}^{2d})} > 1$ for all $i \neq j$. This contradiction proves the claim.

Step 4. Denote by $\mathfrak{L}_0^\perp$ the orthogonal space to $\mathfrak{L}_0$ in $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$. We claim that

$$\|f\|_{L^2(\Omega_{\epsilon_0},\mathbb{R}^{2d})} \lesssim \|Tf\|_H \quad \text{for all } f \in \mathfrak{L}_0^\perp. \tag{56}$$

Indeed, assume the contrary, there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathfrak{L}^\perp$ such that $\|f_n\|_{L^2(\Omega_{\epsilon_0},\mathbb{R}^{2d})} = 1$ and $\|Tf_n\|_H \leq \frac{1}{n}$ for all n. By the same argument as in Step 3, we can find a subsequence,

still denoted by $\{e_j\}_{j=1}^{\infty}$, such that both the sequences $\{T_1 e_j\}_{j=1}^{\infty}$ and $\{T_2 e_j\}_{j=1}^{\infty}$ are Cauchy in H. By Inequality (55) and the fact that $\|Tf_n\|_H \le \frac{1}{n}$, we can conclude that $\{f_n\}_{n=1}^{\infty}$ is also Cauchy in $L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})$. Let $f_0 = \lim_{n\to\infty} f_n$, then $\|Tf_0\|_H = \lim_{n\to\infty} \|Tf_n\|_H = 0$. This implies that $f_0 \in \mathfrak{L}_0$. However, note that $\mathfrak{L}_0^{\perp}$ is closed, as the limit of a sequence of functions in $\mathfrak{L}_0^{\perp}$, f_0 must belong to $\mathfrak{L}_0^{\perp}$. Therefore, we see that $f_0 = 0$. But this contradicts to the fact that $\|f_0\|_{L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})} = \lim_{n\to\infty} \|f_n\|_{L^2(\Omega_{\epsilon_0}, \mathbb{R}^{2d})} = 1$. The claim is proved.

Step 5. From now on, let f be as in (14). We claim that

$$\|Tf\|_H \lesssim \|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}^{\frac{2}{5}} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}.$$

Indeed, for each $\mathfrak{I}_{\alpha_j}$, by Lemma 7.1, we have $\|\mathfrak{I}_{\alpha_j} f\|_{L^\infty(S_-^*\Gamma)} \le C_1 \|f\|_{C^1}^2$. Apply $\mathfrak{I}_{\alpha_j}^{\dagger}$ to both sides and use the fact that $\mathfrak{I}_{\alpha_j}^{\dagger}$ is bounded from L^2 to L^2(see [22]), we obtain

$$\|\mathfrak{N}_{\alpha_j} f\|_{L^2(\Omega,\mathbb{R}^{2d})} \lesssim \|f\|_{C^1(\Omega,\mathbb{R}^{2d})}^2. \tag{57}$$

Then,

$$\begin{aligned}
\|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j),\mathbb{R}^{2d})} &\lesssim \|\mathfrak{N}_{\alpha_j} f\|_{H^3(U(x_j),\mathbb{R}^{2d})}^{\frac{1}{3}} \cdot \|\mathfrak{N}_{\alpha} f\|_{L^2(\Omega,\mathbb{R}^{2d})}^{\frac{2}{3}} && \text{(by interpolation inequality)}\\
&\lesssim \|\mathfrak{N}_{\alpha_j} f\|_{H^3(U(x_j),\mathbb{R}^{2d})}^{\frac{1}{3}} \cdot \|f\|_{C^1(\Omega,\mathbb{R}^{2d})}^{\frac{4}{3}} && \text{(by (57))}\\
&\lesssim \|f\|_{H^3(\Omega,\mathbb{R}^{2d})}^{\frac{1}{3}} \cdot \|f\|_{C^1(\Omega,\mathbb{R}^{2d})}^{\frac{4}{3}} && \text{(by Lemma 6.6, 6.7)}\\
&\lesssim \|f\|_{H^3(\Omega,\mathbb{R}^{2d})}^{\frac{1}{3}} \cdot \|f\|_{H^3(\Omega,\mathbb{R}^{2d})}^{\frac{4}{3}} && \text{(by interpolation inequality)}\\
&= \|f\|_{H^3(\Omega,\mathbb{R}^{2d})}^{\frac{5}{3}}\\
&\lesssim \|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}^{\frac{2}{5}} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}. && \text{(by interpolation inequality)}
\end{aligned}$$

It follows that

$$\|Tf\|_H = \sum_{j=1}^{M} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j),\mathbb{R}^{2d})} \lesssim \|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}^{\frac{2}{5}} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}. \tag{58}$$

This finishes the proof of our claim.

Step 6. Denote by $\mathfrak{L}$ the projection of $\mathfrak{L}_0$ from $L^2(\Omega, \mathbb{R}^{2d})$ to the space $L^2(\Omega, \mathbb{R}^d)$ by taking the last three components. Note that the first three components of f are zero, see (14). Thus the condition $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$ implies that $f \in \mathfrak{L}_0^{\perp}$. Consequently, Inequality (56) holds. Combining this with (58), we see that

$$\|f\|_{L^2(\Omega,\mathbb{R}^{2d})} \lesssim \|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}^{\frac{2}{5}} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}.$$

Therefore, we must have $f = 0$ for $\|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}^{\frac{2}{5}}$ sufficiently small. Finally, note that $\|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})} \lesssim \|c - \tilde{c}\|_{H^{\frac{17}{2}}(\Omega)}$ and that both c and $\tilde{c}$ vanishes near the boundary, we conclude that $f = 0$ implies

$c = \tilde{c}$. This completes the proof of the theorem.

8 Appendix

8.1 Linearization of ODE system

Given the following ODE system:

$$\dot{y} = f(y), \quad y = y_o,$$

where $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. We consider the perturbed the system

$$\dot{y_\epsilon} = f_\epsilon(y_\epsilon), \quad y_\epsilon = y_o,$$

where $f_\epsilon = f + \epsilon g$ with $g \in C^1(\mathbb{R}^d)$. We formally write $y_\epsilon(t) = y(t) + r(t) = y(t) + \epsilon\phi(t) + r_1(t)$, where $\phi(0) = r(0) = r_1(0) = 0$. By substituting $y_\epsilon(t) = y(t) + \epsilon\phi(t) + r_1(t)$ into the perturbed system, we can derive that ϕ satisfies the following equation:

$$\dot{\phi}(t) = \frac{\partial f}{\partial y}(y(t)) \cdot \phi(t) + g(y(t)), \quad \phi(0) = 0. \tag{59}$$

By Grownwall's inequality, we can show that $|r(t)| \leq C\epsilon$ and $|r_1(t)| \leq C\epsilon^2$, where C is a constant depending on $\|f\|_{C^2} + \|g\|_{C^1}$.

We solve equation (59) as follows. Let $A(t) = \frac{\partial f}{\partial y}(y(t))$. Let $\Phi(t)$ and $\Psi(t)$ be the solution to the following ODE system:

$$\dot{\Phi}(t) = -\Phi(t)A(t), \quad \Phi(0) = Id;$$
$$\dot{\Psi}(t) = A(t)\Psi(t), \quad \Psi(0) = Id.$$

A straightforward calculation shows that $\Phi(t)\Psi(t) \equiv \Phi(0)\Psi(0) = Id$. Moreover,

$$\phi(t) = \Phi(t)^{-1} \int_0^t \Phi(s)g(y(s))\, ds.$$

8.2 Proof of Lemma 4.2 and Lemma 4.4

Proof of 4.2. We only show (27), since (28) follows in a similar way. For simplicity, denote $D = (3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)$. We first show that

$$\hat{g}(t, y, \lambda) = \hat{g}_*(t, y, \lambda) + O(\frac{1}{\sqrt{\lambda}}) \quad \text{in} \quad L^2(D). \tag{60}$$

Indeed, by direct calculation,

$$\hat{g}(t, y, \lambda) - \hat{g}_*(t, y, \lambda) = (a(t) - a(t_1))e^{i\lambda\hat{\tau}_*} + a(t)(e^{i\lambda\hat{\tau}} - e^{i\lambda\hat{\tau}_*}). \tag{61}$$

It suffices to show that

$$
\begin{aligned}
R_1 &:= \|(a(t)-a(t_1))e^{i\lambda\hat{\tau}_*}\|_{L^2(D)} \lesssim \frac{1}{\sqrt{\lambda}},\\
R_2 &:= \|a(t)(e^{i\lambda\hat{\tau}}-e^{i\lambda\hat{\tau}_*})\|_{L^2(D)} \lesssim \frac{1}{\sqrt{\lambda}}.
\end{aligned}
$$

We first estimate R_1. By Lemma 3.1 in [5], we have $|a(t)| \approx \lambda^{\frac{d}{4}}$. By equation (23), we further derive that $|\dot{a}(t)| \approx \lambda^{\frac{d}{4}}$, thus

$$
a(t)-a(t_1) = \int_0^1 \dot{a}(t_1+s(t-t_1))\, ds(t-t_1) = O(\lambda^{\frac{d}{4}})|t-t_1|.
$$

Therefore,

$$
\|(a(t)-a(t_1))e^{i\lambda\hat{\tau}_*}\|_{L^2(D)}^2 \lesssim \int_D \lambda^{\frac{d}{2}}(t-t_1)^2 e^{-\lambda(t-t_1,y)\Im\hat{M}(t_1)(t-t_1,y)^\dagger}\, dtdy \lesssim \frac{1}{\lambda}.
$$

This proves $R_1 \lesssim \frac{1}{\sqrt{\lambda}}$.

We next estimate R_2. Write $\hat{\tau} = \hat{\tau}_* + \delta\hat{\tau}$, then $\delta\hat{\tau} = O(|(t-t_1,y)|^3)$ and hence $|1-e^{i\lambda\delta\hat{\tau}}| \lesssim \lambda\cdot O(|(t-t_1,y)|^3)$. It follows that

$$
R_2 \le \int_D |a(t)e^{i\lambda\hat{\tau}_*}|^2\cdot|1-e^{i\lambda\delta\hat{\tau}}|\, dtdy \lesssim \int_D \lambda^{\frac{d}{2}}\cdot\lambda\cdot|(t-t_1,y)|^3 e^{-2\lambda(t-t_1,y)\Im\hat{M}(t_1)(t-t_1,y)^\dagger}\, dtdy \lesssim \frac{1}{\lambda}.
$$

This completes the proof of (60).

We now proceed to show (27). By direct calculation,

$$
\begin{aligned}
\frac{\partial\hat{g}}{\partial y} - \frac{\partial\hat{g}_*}{\partial y} &= i\lambda\frac{\partial\hat{\tau}}{\partial y}\cdot\hat{g} - i\lambda\frac{\partial\hat{\tau}_*}{\partial y}\cdot\hat{g}_*\\
&= i\lambda(\frac{\partial\hat{\tau}}{\partial y} - \frac{\partial\hat{\tau}_*}{\partial y})\cdot\hat{g} + i\lambda\frac{\partial\hat{\tau}_*}{\partial y}\cdot(\hat{g}-\hat{g}_*)
\end{aligned}
$$

One can check that $\frac{\partial\hat{\tau}}{\partial y} - \frac{\partial\hat{\tau}_*}{\partial y} = O|(t-t_1,y)|^2$, then a similar argument as used in the estimate of R_1 above shows that

$$
\|\lambda(\frac{\partial\hat{\tau}}{\partial y} - \frac{\partial\hat{\tau}_*}{\partial y})\cdot\hat{g}\|_{L^2(D)}^2 \lesssim 1.
$$

Besides, (60) implies that

$$
\|\lambda\frac{\partial\hat{\tau}_*}{\partial y}\cdot(\hat{g}-\hat{g}_*)\|_{L^2(D)}^2 \lesssim \lambda.
$$

Combining these two estimates together, we conclude that

$$
\|\frac{\partial\hat{g}}{\partial y} - \frac{\partial\hat{g}_*}{\partial y}\|_{L^2(D)}^2 \lesssim \lambda.
$$

Similarly, we can show that

$$
\|\frac{\partial\hat{g}}{\partial t} - \frac{\partial\hat{g}_*}{\partial t}\|_{L^2(D)}^2 \lesssim \lambda.
$$

This completes the proof of (27) and hence the lemma.

Proof of Lemma 4.4. Denote $D = (3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)$ again. We first show (30). Since x is restricted to $V(x_1) \subset \Gamma$, it suffices to show that

$$\hat{g}^-(t, y, \lambda) + \hat{g}(t, y, 0, \lambda) = O(\sqrt{\lambda}) \quad \text{in} \quad H^1(D).$$

But this is a direct consequence of Lemma 4.2 and the fact that $\hat{g} = -\hat{g}_*$.

We now prove (31). By direct calculate

$$\begin{aligned}
\frac{\partial g}{\partial \nu}(t, x) &= \frac{\partial}{\partial \nu}(a(t)e^{i\lambda\tau(t,x)}) = i\lambda g \cdot \frac{\partial \tau}{\partial \nu} \\
&= i\lambda g \cdot \langle \xi(t) + M(t)(x - x(t), \nu(x)\rangle \\
&= i\lambda g \cdot \langle \xi(t_1), \nu(x_1)\rangle + i\lambda g \cdot (\langle \xi(t), \nu(x)\rangle - \langle \xi(t_1), \nu(x_1)\rangle) + i\lambda g \cdot \langle M(t)(x - x(t), \nu(x)\rangle.
\end{aligned}$$

Note that in the coordinate $x = F(y)$,

$$\begin{aligned}
|\langle M(t)(x - x(t)), \nu(x)\rangle| &= O(|(t - t_1, y)|), \\
|\langle \xi(t), \nu(x)\rangle - \langle \xi(t_1), \nu(x_1)\rangle| &= O(|(t - t_1, y)|).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|g \cdot (\langle \xi(t), \nu(x)\rangle - \langle \xi(t_1), \nu(x_1)\rangle)\|^2_{L^2(D)} &\lesssim \frac{1}{\lambda}, \\
\|g \cdot \langle M(t)(x - x(t), \nu(x)\rangle\|^2_{L^2(D)} &\lesssim \frac{1}{\lambda}.
\end{aligned}$$

Thus

$$\frac{\partial g}{\partial \nu}(t, x) = i\lambda g \cdot \langle \xi(t_1), \nu(x_1)\rangle + O(\sqrt{\lambda}).$$

Similarly,

$$\frac{\partial g^-}{\partial \nu}(t, x) = i\lambda g^- \cdot \langle \xi^-(t_1), \nu(x_1)\rangle + O(\sqrt{\lambda}).$$

Finally, using (30) and the fact that $\langle \xi^-(t_1), \nu(x_1)\rangle = -\langle \xi(t_1), \nu(x_1)\rangle$, we conclude that (31) holds. This completes the proof of the lemma.

8.3 An estimate on Gaussian beam interactions

Lemma 8.1. *Assume that M_1 and M_2 are two symmetric positive definite matrices such that $0 < c_0 < M_1, M_2 < c_1$, and N_1 and N_2 are two symmetric matrices such that $\|N_1\|, \|N_2\| \leq c_2$. Let $\delta x, \delta \xi$ be two vectors in $\mathbb{R}^d$ and $\lambda \gg 1$. Then exists $c_3 > 0$ depending only on c_0, c_1 and c_2 such that*

$$|\int_{\mathbb{R}^d} e^{i\lambda \cdot \langle \delta\xi, x\rangle - \lambda x^T(M_1 + i\cdot N_1)x - \lambda(x - \delta x)^T(M_2 + i\cdot N_2)(x - \delta x)}| \lesssim \frac{1}{\lambda^{\frac{d}{2}}} e^{-c_3\lambda(|\delta x|^2 + |\delta\xi|^2)}.$$

Proof. See Lemma 3.7 in [5].

References

[1] G. Alessandrini and J. Sylvester. Stability for a multidimensional inverse spectral theorem, Comm.PDE, 15(1990), 711-736.

[2] R. Alexandre, J. Akian and S. Bougacha. Gaussian beams summation for the wave equation in a convex domain, Commun. Math. Sci. 7(4)(2009), 973-1008.

[3] V. I. Arnold. Singularity Theory, London Mathematical Society Lecture Note Series(53), Cambrige Univeristy Press, 1981.

[4] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. Singularites of Differentiable Maps. I, volume 82 of monographs in mathematics. Birkh*ä*user, Boston, 1985.

[5] G. Bao, J. Qian, L. Ying and H. Zhang. A convergent multiscale Gaussian-beam parametrix for wave equations, Comm.PDE, accepted.

[6] G. Bao and K. Yun. On the stability of an inverse problem for the wave equation, Inverse Problems, 25(4)(2009), R1-R7.

[7] M. Belishev. An approach to multidimensional inverse problems for the wave equation, Dokl. Akad. Nauk. SSR 297(1987), 524-527.

[8] M. Belishev. Boundary control in reconstruction of manifolds and metrics(the BC method), Inverse Problems 13(1997), R1-R45.

[9] M. Belishev. Recent progress in the boundary control method, Inverse Problems 23(2007), R1-R67.

[10] M. Belishev and Y. V. Kurylev. To the reconstruction of a Riemannian manifold via its spectral data(BC-method), Comm.P.D.E 17(1992), 765-804.

[11] M. Bellassoued and D. D. S. Ferreira. Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map, Inverse Problems and Imaging, 5(4)(2011), 745-773.

[12] A. Greenleaf and A. Seeger. Fourier integral operators with cusp singularites, American Journal of Mathematics, 120(5)(1998), 1077-1119.

[13] L. Hömander. The Analysis of Linear Partial Differential Operators, III and IV, volume 274 and 275 of Grundelehren der Mathematischen Wissenschaften, Spring-Verlag, Berlin, 1985.

[14] V. Isakov. Inverse Problems for Partial Differential Equations, volume 127 of Applied Mathematical Sciences, second edition, Springer Science+Business Media, Inc, New York, 2006.

[15] A. Katchalov, Y. Kurylev and M. Lassas. Inverse Boundary Spectral Problems. Chapman Hall/CRC, Boca Raton, 2001.

[16] I. Lasiecka, J. L. Lions and R. Triggiani. Non homogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures et Appl., 65(1986), 149-192.

[17] J. M. Lee. Riemannian manifolds, an introduction to curvature, volume 176 of Graduate Texts in Mathematics, Spring-Verlag, New York, 1997.

[18] J. Qian and L. Ying. Fast multiscale Gaussian wavepacket transforms and multiscale Gaussian beams for the wave equation. SIAM. MMS, 8 (2010).

[19] L. Pestov and G. Uhlmann. The scattering relation and the Dirichlet-to-Neumann map, Contemporary Math, 412(2006), 249-262.

[20] J. Ralston. Gassian beams and the propagation of singularities, Studies in Partial Differential Equations, MAA Studies in Mathematics, 23(1983), Edited by W. Littman, 206-248.

[21] V. G. Romanov. Invere Problems of Mathematical Physics. VNU Science Press, Utreht, the Netherlands, 1987.

[22] V. A. Sharafutdinov. Ray Transform on Riemannian Manifolds. Lecture notes, University of Oulu, 1999.

[23] P. Stefanov and G. Uhlmann. Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media, J. Funct. Anal., 154(2)(1998), 330-358.

[24] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity Duke Math. J., 123(2004), 445-467.

[25] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics, Journal Amer. Math. Soc. 18(2005), 975-1003.

[26] P. Stefanov and G. Uhlmann. Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map, IMRN 17(2005), 1047-1061.

[27] P. Stefanov and G. Uhlmann. The X-Ray transform for a generic family of curves and weights J. Geom. Anal. 18(1)(2008), 81-97

[28] P. Stefanov and G. Uhlmann. Integral geometry of tensor fields on a class of non-simple Riemannian manifolds, American J. of Math., 130 (2008), 239-268.

[29] P. Stefanov and G. Uhlmann. Linearizing non-linear inverse problems and its applications to inverse backscattering, Journal Functional Analysis 256 (2009), 2842-2866.

[30] P. Stefanov and G. Uhlmann. Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds, Journal Diff. Geometry 82 (2009), 383-409.

[31] Z. Sun. On continuous dependence for an inverse initial boundary value problem for the wave equation, J. Math. Anal. Appl, 150(1990), 188-204.

[32] G. Uhlmann. The cauchy data and the scattering relation. IMA Publications, Springer-Verlag, 137(2003), 263-288.

[33] P. Stefanov and G. Uhlmann. The geodesic X-ray transform with fold caustics, Anal. and PDE, to appear,

[34] C. Montalto. Stable determination of a simple metric, a covector field and a potential from the hyperbolic Dirichlet-to-Nuemann map, ArXiv, 2012.